Relative Hochschild Cohomology and its Gerstenhaber Product

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Outline

Why do we care?

- Homology is a **useful tool** in studying algebraic objects: it **encodes meaningful information** and provides insight into their properties and structure(s).
- Hochschild cohomology encodes information on **deformations**, **smoothness** and **representations** of algebras, among others.
- When Hochschild cohomology is **finitely generated**, it realizes a structure on support varieties in which geometers are extremely interested.

Algebras over a ring (I)

Definition

Let k be an associative commutative ring. We say that A is a k *algebra* if it is a k module and a ring, where the product $\mu : A \times A \longrightarrow A$ is bilinear.

Examples:

- Commutative: $k[x]$, $k[x_1, \ldots, x_n]$, $k[x]/(x^n)$ for $n \in \mathbb{N}$.
- Noncommutative: $k\langle x, y\rangle/(yx - xy - x^2)$.

Definition

Let A be a k algebra. We define A^{op} the *opposite algebra of* A as the vector space A with multiplication $\mu_{op} : A \times A \longrightarrow A$ given by:

$$\mu_{op}(a, b) = \mu(b, a) \text{ for all } a, b \in A.$$

Algebras over a ring (and II)

Definition

Let A be a k algebra. We define A^e the *enveloping algebra of A* as the vector space $A \otimes A^{op}$ with multiplication $\mu^e : A^e \times A^e \longrightarrow A^e$ given by:

$$\mu^e((a_1 \otimes b_1), (a_2 \otimes b_2)) = \mu(a_1, a_2) \otimes \mu_{op}(b_1, b_2) = a_1 a_2 \otimes b_2 b_1$$

for all $a_1, a_2, b_1, b_2 \in A$.

Examples:

- $k[x]^e = k[x] \otimes k[y] \cong k[x, y]$.
- $k[x]/(x^n)^e = k[x]/(x^n) \otimes k[y]/(y^n) \cong k[x, y]/(x^n, y^n)$ for $n \in \mathbb{N}$.

Hochschild Cohomology

Definition

The *Hochschild cohomology* of a k algebra A with coefficients in a left A^e module M is $HH^\bullet(A,M) = \bigoplus_{n\in\mathbb{N}} HH^n(A,M)$, where for $n \in \mathbb{N}$:

$$HH^n(A,M) = \mathrm{Ext}^n_{A^e}(A,M).$$

Hence to compute the Hochschild cohomology we need A^e projective resolutions of A. We now provide a canonical one.

Special modules and bimodules over an algebra

Remark

A k algebra A is a left A^e module under:

$$(a \otimes b) \cdot c = acb \text{ for all } a, b, c \in A.$$

In particular $HH^\bullet(A) := HH^\bullet(A, A)$ is well defined.

Remark

The tensor product $A^{\otimes n} = A \otimes \overset{(n)}{\cdots} \otimes A$ is is a left A^e module under:

$$(a \otimes b) \cdot (c_1 \otimes c_2 \cdots \otimes c_{n-1} \otimes c_n) = ac_1 \otimes c_2 \cdots \otimes c_{n-1} \otimes c_n b$$

for all $a, b, c_1, \ldots, c_n \in A$.

The Bar resolution

Consider the sequence of left A^e modules:

$$\cdots \xrightarrow{d_3} A^{\otimes 4} \xrightarrow{d_2} A^{\otimes 3} \xrightarrow{d_1} A \otimes A \xrightarrow{\mu} A \longrightarrow 0$$

with:

$$d_n(a_0 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}$$

for all $a_0, \ldots, a_{n+1} \in A$. This is a complex by direct computation. It has a contracting homotopy $s_n : A^{\otimes(n+2)} \longrightarrow A^{\otimes(n+3)}$:

$$s_n(a_0 \otimes \cdots \otimes a_{n+1}) = 1 \otimes a_0 \otimes \cdots \otimes a_{n+1}$$

so the complex is exact. Moreover since $A^{\otimes n} \cong \bigoplus_{i \in I} k\alpha_i$ as k modules:

$$A^{\otimes(n+2)} \cong A^e \otimes A^{\otimes n} \cong \bigoplus_{i \in I} A^e(1 \otimes 1 \otimes \alpha_i)$$

so $A^{\otimes(n+2)}$ are free A^e modules, and the complex is a free resolution.

Cochains in the Bar resolution (or how to compute Ext)

Given:

$$\cdots \xrightarrow{d_3} A^{\otimes 4} \xrightarrow{d_2} A^{\otimes 3} \xrightarrow{d_1} A \otimes A \longrightarrow 0$$

apply $\mathrm{Hom}_{A^e}(-, M)$:

$$0 \longrightarrow \mathrm{Hom}_{A^e}(A \otimes A, M) \xrightarrow{d_1^*} \mathrm{Hom}_{A^e}(A^{\otimes 3}, M) \xrightarrow{d_2^*} \mathrm{Hom}_{A^e}(A^{\otimes 4}, M) \xrightarrow{d_3^*} \cdots$$

using $\mathrm{Hom}_{A^e}(A^{\otimes(n+2)}, M) \cong \mathrm{Hom}_k(A^{\otimes n}, M)$ we obtain:

$$0 \longrightarrow \mathrm{Hom}_k(k, A) \xrightarrow{d_1^*} \mathrm{Hom}_k(A, M) \xrightarrow{d_2^*} \mathrm{Hom}_k(A \otimes A, M) \xrightarrow{d_3^*} \cdots$$

which can still compute $HH^\bullet(A, M)$. We are interested in $M = A$.

Definition

The elements of $\mathrm{Hom}_k(A^{\otimes n}, M)$ are called *Hochschild cochains with coefficients in M*.

Cup product at the cochain level

Definition

Let $f \in \mathrm{Hom}_k(A^{\otimes m}, A)$ and $g \in \mathrm{Hom}_k(A^{\otimes n}, A)$. *The cup product* $f \smile g$ is the element of $\mathrm{Hom}_k(A^{\otimes(m+n)}, A)$ given by:

$$(f \smile g)(a_1 \otimes \cdots \otimes a_{m+n}) = f(a_1 \otimes \cdots \otimes a_m)g(a_{m+1} \otimes \cdots \otimes a_{m+n})$$

for all $a_1, \ldots, a_{m+n} \in A$. If $m = 0$ this is interpreted as:

$$(f \smile g)(a_1 \otimes \cdots \otimes a_{m+n}) = f(1)g(a_1 \otimes \cdots \otimes a_n)$$

and similarly if $n = 0$.

Properties of the cup product

Proposition

Let $f \in \mathrm{Hom}_k(A^{\otimes m}, A)$, $g \in \mathrm{Hom}_k(A^{\otimes n}, A)$, and $h \in \mathrm{Hom}_k(A^{\otimes l}, A)$.

1. The cup product is associative:

$$(f \smile g) \smile h = f \smile (g \smile h).$$

2. It satisfies:

$$d^*_{m+n+1}(f \smile g) = (d^*_{m+1}(f)) \smile g + (-1)^m f \smile (d^*_{n+1}(g)).$$

Theorem

- The cup product on $HH^\bullet(A)$ is *graded associative*.
- The cup product on $HH^\bullet(A)$ is *graded commutative*.
- The cup product on cochains forms a *differential graded algebra*.

Gerstenhaber bracket

Definition

Let $f \in \mathrm{Hom}_k(A^{\otimes m}, A)$ and $g \in \mathrm{Hom}_k(A^{\otimes n}, A)$. *The Gerstenhaber bracket* $[f, g]$ is the element of $\mathrm{Hom}_k(A^{\otimes(m+n-1)}, A)$ given by:

$$[f, g] = f \circ g - (-1)^{(m-1)(n-1)} g \circ f$$

where the *circle product* is given by:

$$(f \circ g)(a_1 \otimes \cdots \otimes a_{m+n-1}) = \\ = \sum_{i=1}^{m} (-1)^u f(a_1 \otimes \overset{(i-1)}{\cdots} \otimes g(a_i \otimes \cdots \otimes a_{i+n-1}) \otimes \overset{(m-i)}{\cdots} \otimes a_{m+n-1})$$

where $u = (n-1)(i-1)$, for all $a_1, \ldots, a_{m+n-1} \in A$. If $m = 0$ then $f \circ g = 0$ and if $n = 0$ then $g(1)$ replaces $g(a_i \otimes \cdots \otimes a_{i+n-1})$.

Properties of the Gerstenhaber bracket

Proposition

1. The Gerstenhaber bracket is graded anti-commutative.
2. The Gerstenhaber bracket satisfies the graded Jacobi identity.
3. The Gerstenhaber bracket on cochains forms a *differential graded Lie algebra*.

Proposition

Let $\alpha \in HH^m(A)$, $\beta \in HH^n(A)$, and $\gamma \in HH^l(A)$, then:

$$[\alpha \smile \beta, \gamma] = [\alpha, \gamma] \smile \beta + (-1)^{m(l-1)} \alpha \smile [\beta, \gamma].$$

Theorem

- $(HH^\bullet(A), \smile, [-,-])$ is a *Gerstenhaber algebra*.

Relative exact sequences

Definition

Let R be a ring, a sequence of R modules:

$$\cdots \longrightarrow C_i \xrightarrow{t_i} C_{i-1} \longrightarrow \cdots$$

is called *exact* if $\mathrm{Im}(t_i) = \mathrm{Ker}(t_{i-1})$.

Definition

Let $1_R \in S \subseteq R$ a subring. An exact sequence of R modules is called (R, S) *exact* if $\mathrm{Ker}(t_i)$ is a direct summand of C_i as an S module.

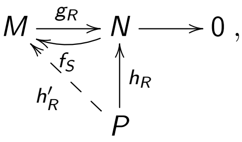
This is equivalent to the sequence splitting, and to the existence of an S homotopy (hence the sequence is exact as S modules).

Relative projective modules

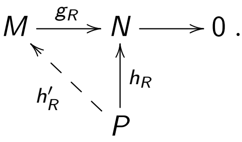
Definition

An R module P is said to be (R, S) *projective* if, for every (R, S) exact sequence $M \xrightarrow{g} N \longrightarrow 0$ and every R homomorphism $h : P \longrightarrow N$, there is an R homomorphism $h' : P \longrightarrow M$ with $gh' = h$.

Relative:



Usual:



Cup product on tensor product of complexes (I)

Consider $P_\bullet$ an A^e projective resolution of A. It can be proven that the total complex of $P_\bullet \otimes_A P_\bullet$ is also an A^e projective resolution of A. Moreover, it can be proven that there exists a diagonal map $\Delta : P_\bullet \longrightarrow \mathrm{Tot}(P_\bullet \otimes_A P_\bullet)$ lifting the identity map on A.

Definition

Let $P_\bullet$ an A^e projective resolution of A, $f \in \mathrm{Hom}_{A^e}(P_m, A)$, and $g \in \mathrm{Hom}_{A^e}(P_n, A)$. *The cup product* $f \smile g$ is defined by $f \smile g = \mu(f \otimes g)\Delta$.

This definition can be proven to be equivalent to the previous cup product.

Cup product on tensor product of complexes (and II)

To prove all these claims, we use:

1. the characterization of free modules over a ring,
2. the characterization of projective modules over a ring,
3. the Künneth Theorem,
4. the Comparison Theorem.

We want analogues of this results in relative homological algebra.

Ongoing research

- Are there analogous results and characterizations in *relative* homological algebra? **Yes**.
- Does *relative* Hochschild cohomology have a cup product? If it has more than one, are they equivalent?
- Does it have a Gerstenhaber bracket? Does it induce a structure of Gerstenhaber algebra?

Something to take home

- Hochschild cohomology is intimately related with the Ext functor.
- Over a field this is well behaved and fairly well understood.
- There is a lot of progress to be made in natural generalizations.

Thank you!

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