# An Approach to Determinants

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# Outline

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# Why do we care?

- Most of the times determinants are given as a **formula** without any kind of explanation, for example as an expansion by rows or columns.
- **That is wrong**, basic Mathematics should be understood as deep as possible to allow a solid base over which we can build knowledge.
- Linear Algebra is one of the most basic and **powerful tools** a mathematician has. Understanding determinants from first principles enable us to see how elementary results allow deep understanding of complex concepts.

# Remarks

- We will be working over $\mathbb{R}$, but most (if not all) of the following can be generalized to any commutative ring $R$.
- Let $S_n$ be the group of permutations of $n$ elements. Then $A_n = \{\sigma \in S_n | \mathrm{sgn}(\sigma) = 1\}$ and for any transposition $\tau \in S_n$ we have $S_n \setminus A_n = \{\sigma\tau | \sigma \in A_n\}$.

# Notation

- Given $A \in M_n(\mathbb{R})$ a matrix, we will denote its columns by $C_1, \dots, C_n$.
- The elementary matrices are:
  - $D_n(i, \lambda)$, the matrix obtained by multiplying the $i$th row of $1_n$ by $\lambda \in \mathbb{R} \setminus \{0\}$,
  - $P_n(i, j)$, the matrix obtained by exchanging the $i$th and $j$th rows $(i \neq j)$ of $1_n$,
  - $E_n(i, j, \mu)$, the matrix obtained by adding to the $i$th row of $1_n$ the $j$th row $(i \neq j)$ of $1_n$ multiplied by $\mu \in \mathbb{R}$.

# Results

The price we pay for working with first principles is a heavy use of the structure of matrices.

## Theorem (PAQ-reduction)

Given any $A \in M_{m\times n}(\mathbb{R})$, there exist $P \in M_m(\mathbb{R})$ and $Q \in M_n(\mathbb{R})$ invertible (in fact product of elementary matrices) such that:

$$PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \text{ and } r \text{ does not depend on } P \text{ nor } Q.$$

## Corollary

A matrix $A \in M_n(\mathbb{R})$ is invertible if and only if it is a product of elementary matrices.

## Corollary

A matrix $A \in M_n(\mathbb{R})$ is invertible if and only if its rank is $n$.

# First principles

**Definition**

A *determinant* is a map $\det : M_n(\mathbb{R}) \longrightarrow \mathbb{R}$ satisfying:

1. it is linear with respect to each column,
2. is alternating,
3. $\det(1_n) = 1$.

With this definition we need to show that such a map exists, and hopefully that it is unique.

# In matrix notation

That is, given $C_1, \dots, C_n, C_j' \in M_{n\times 1}(\mathbb{R})$, $\alpha \in \mathbb{R}$ we want by linearity:

1. $\det(C_1, \dots, C_j + C_j', \dots, C_n) =$
   $\det(C_1, \dots, C_j, \dots, C_n) + \det(C_1, \dots, C_j', \dots, C_n)$,
2. $\det(C_1, \dots, \alpha C_j, \dots, C_n) = \alpha \det(C_1, \dots, C_j, \dots, C_n)$,

if $C_i = C_j$ for some $1 \leq i < j \leq n$ then for alternating:

3. $\det(C_1, \dots, C_i, \dots, C_j, \dots, C_n) = 0$,

and always:

4. $\det(1_n) = 1$,

# First properties (I)

## Proposition

Let $\det : M_n(\mathbb{R}) \longrightarrow \mathbb{R}$ be a determinant. Let $C_1, \ldots, C_n \in M_{n\times 1}(\mathbb{R})$, then:

$$\det(C_1, \ldots, C_i, \ldots, C_j, \ldots, C_n) = -\det(C_1, \ldots, C_j, \ldots, C_i, \ldots, C_n)$$

that is, exchanging two columns changes the sign of the determinant.

That is, determinants should be antisymmetric. In fact, we prove that every alternating multilinear map is antisymmetric.

# First properties (II)

## Proof.

We have:

$$\begin{aligned} 0 &= \det(C_1, \ldots, C_i + C_j, \ldots, C_j + C_i, \ldots, C_n) \\ &= \det(C_1, \ldots, C_i, \ldots, C_j, \ldots, C_n) + \det(C_1, \ldots, C_i, \ldots, C_i, \ldots, C_n) \\ &+ \det(C_1, \ldots, C_j, \ldots, C_j, \ldots, C_n) + \det(C_1, \ldots, C_j, \ldots, C_i, \ldots, C_n) \\ &= \det(C_1, \ldots, C_i, \ldots, C_j, \ldots, C_n) + \det(C_1, \ldots, C_j, \ldots, C_i, \ldots, C_n) \end{aligned}$$

by applying alternating, linearity and alternating again. Hence:

$$-\det(C_1, \ldots, C_j, \ldots, C_i, \ldots, C_n) = \det(C_1, \ldots, C_i, \ldots, C_j, \ldots, C_n).$$

□

# First properties (III)

## Proposition

Let $\det : M_2(\mathbb{R}) \longrightarrow \mathbb{R}$ be a determinant. Then:

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

So in particular at most one determinant exists in dimension two, and it must have this form.

# First properties (and IV)

## Proof.

We have:

$$
\begin{aligned}
\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} &= a \det \begin{bmatrix} 1 & b \\ 0 & d \end{bmatrix} + c \det \begin{bmatrix} 0 & b \\ 1 & d \end{bmatrix} \\
&= a \left( b \det \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + d \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \\
&+ c \left( b \det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d \det \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right) = ad - cb
\end{aligned}
$$

where we have used linearity on the first column, then on the second column, and finally alternating, antisymmetric and $\det(1_2) = 1$. □

# Determinant of elementary matrices (I)

## Proposition

Let $\det : M_n(\mathbb{R}) \longrightarrow \mathbb{R}$ be a determinant. Then:

1. $\det(D_n(i, \lambda)) = \lambda$,
2. $\det(P_n(i, j)) = -1$,
3. $\det(E_n(i, j, \mu)) = 1$.

# Determinant of elementary matrices (and II)

## Proof.

1. $\det(D_n(i,\lambda)) = \lambda \det(1_n) = \lambda$ by multilinearity,
2. $\det(P_n(i,j)) = -\det(1_n) = -1$ by antisymmetric,
3. $$\det(E_n(i,j,\mu)) = \det(1_n) + \mu \det \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & 1 & \\ & & & \ddots & & \\ & & 0 & & 0 & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix} = 1$$

   by linearity and alternating.

□

# Determinant of a product of matrices (I)

## Proposition

Let $\det : M_n(\mathbb{R}) \longrightarrow \mathbb{R}$ be a determinant. Then for all $A, B \in M_n(\mathbb{R})$ we have:

$$\det(AB) = \det(A)\det(B).$$

In other words, a determinant is multiplicative.

# Determinant of a product of matrices (II)

**Proof.**

Let $C_1, \ldots, C_n$ be the columns of $A$. We first check the claim for $B$ an elementary matrix:

1. $\det(AD_n(i,\lambda)) = \det(C_1, \ldots, \lambda C_i, \ldots, C_n) = \lambda \det(A) = \det(A)\det(D_n(i,\lambda))$,
2. $\det(AP_n(i,j)) = \det(C_1, \ldots, C_j, \ldots, C_i, \ldots, C_n) = -\det(A) = \det(A)\det(P_n(i,j))$,
3. $\det(AE_n(i,j,\mu)) = \det(C_1, \ldots, C_j + \mu C_i, \ldots, C_n) = \det(A) + \mu\det(C_1, \ldots, C_i, \ldots, C_i, \ldots, C_n) = \det(A) = \det(A)\det(E_n(i,j,\mu))$.

# Determinant of a product of matrices (III)

## Proof.

If $B = E_1 \dots E_m$ is a product of elementary matrices, then by induction on $m$ (the case $m = 1$ is what we just proved):

$$\begin{aligned} \det(AB) &= \det((AE_1 \cdots E_{m-1})E_m) \\ &= \det(AE_1 \dots E_{m-1}) \det(E_m) \\ &= \det(A) \det(E_1 \dots E_{m-1}) \det(E_m) \\ &= \det(A) \det(E_1 \dots E_m) = \det(A) \det(B) \end{aligned}$$

using that $E_m$ is an elementary matrix and induction hypothesis. This also yields that $\det(B) = \det(E_1) \dots \det(E_n) \neq 0$.

# Determinant of a product of matrices (and IV)

**Proof.**

If $B$ is not a product of elementary matrices, then it is not invertible, so it has rank $r$ less than $n$. By the $PAQ$-reduction of $B$, we know that there exists $Q \in M_n(\mathbb{R})$ product of elementary matrices (so $\det(Q) \neq 0$) such that $BQ = (C'_1, \dots, C'_r, 0, \dots, 0)$. In particular $\det(BQ) = 0$ since at least one column is all zeroes. By the previous case $\det(BQ) = \det(B)\det(Q)$, and thus $\det(B) = 0$.
Consider now $ABQ = (C''_1, \dots, C''_r, 0, \dots, 0)$, we analogously have
$0 = \det(ABQ) = \det(AB)\det(Q)$ and thus
$\det(AB) = 0 = \det(A)\det(B)$. □

# Powerful conclusions (I)

In fact in the above reasoning we have proven:

**Theorem**

Let $\det : M_n(\mathbb{R}) \longrightarrow \mathbb{R}$ be a determinant. Then $A \in M_n(\mathbb{R})$ is invertible if and only if $\det(A) \neq 0$.

Moreover, given $A, B \in M_n(\mathbb{R})$ with $AB = 1_n$ then $A$ and $B$ are invertible since $\det(A)\det(B) = 1$, and thus $B^{-1} = A$.

# Powerful conclusions (II)

**Theorem**

Let $\det : M_n(\mathbb{R}) \longrightarrow \mathbb{R}$ be a determinant. Then for all $A \in M_n(\mathbb{R})$ we have $\det(A) = \det(A^T)$.

That is, the properties of the determinant established for the rows of a matrix also hold for the columns of that matrix.

# Powerful conclusions (and III)

## Proof.

If $A$ is invertible, then it can be written as the product of elementary matrices $A = E_1 \dots E_m$. Since $A^T = E_m^T \dots E_1^T$, it is enough to prove that $\det(E_i) = \det(E_i^T)$. That is true since $D_n(i\lambda)^T = D_n(i,\lambda)$, $P_n(i,j)^T = P_n(i,j)$, $E_n(i,j,\mu)^T = E_n(j,i,\mu)$ and $\det(E_n(i,j,\mu)^T) = 1 = \det(E_n(j,i,\mu))$.
If $A$ is not invertible then $A^T$ is not invertible and $\det(A^T) = 0 = \det(A)$. □

# Uniqueness

**Theorem**

Let $\det, \det' : M_n(\mathbb{R}) \longrightarrow \mathbb{R}$ be two determinants. Then $\det(A) = \det'(A)$ for all $A \in M_n(\mathbb{R})$, so $\det = \det'$.

So if it exists, the determinant is unique.

**Proof.**

We know that both det and $\det'$ take the same values over the elementary matrices, and hence over all the invertible matrices. Moreover, they are both zero over the non invertible matrices. They are thus equal. □

# Existence (I)

## Theorem

Given any $A = (a_{ij}) \in M_n(\mathbb{R})$, define $\det, \det' : M_n(\mathbb{R}) \longrightarrow \mathbb{R}$ as:

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{\sigma(1)1} \cdots a_{\sigma(n)n}.$$

Then det is a determinant.

So a determinant exists.

# Existence (II)

## Proof.

We just need to check the properties of the definition. Consider columns:

$$C_k = \begin{bmatrix} a_{1k} & \cdots & a_{nk} \end{bmatrix}^T \text{ for } 1 \leq k \leq n \text{ and } C_j' = \begin{bmatrix} a_{1j}' & \cdots & a_{nj}' \end{bmatrix}^T,$$

then:

$$\begin{aligned}
& \det(C_i, \ldots, C_j + C_j', \ldots, C_n) \\
= & \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{\sigma(1)1} \cdots (a_{\sigma(j)j} + a_{\sigma(j)j}') \cdots a_{\sigma(n)n} \\
= & \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{\sigma(1)1} \cdots a_{\sigma(j)j} \cdots a_{\sigma(n)n} \\
+ & \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{\sigma(1)1} \cdots a_{\sigma(j)j}' \cdots a_{\sigma(n)n} \\
= & \det(C_i, \ldots, C_j, \ldots, C_n) + \det(C_i, \ldots, C_j', \ldots, C_n).
\end{aligned}$$

# Existence (III)

**Proof.**

For any $\alpha \in \mathbb{R}$ we have:

$$\det(C_i, \dots, \alpha C_j, \dots, C_n) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) a_{\sigma(1)1} \cdots \alpha a_{\sigma(j)j} \cdots a_{\sigma(n)n}$$

$$= \alpha \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) a_{\sigma(1)1} \cdots a_{\sigma(j)j} \cdots a_{\sigma(n)n} = \alpha \det(C_i, \dots, C_j, \dots, C_n).$$

Let $C_i = C_j$ for $1 \leq i < j \leq n$, so in particular $a_{\sigma(i)i} = a_{\sigma(i)j}$ and $a_{\sigma(j)i} = a_{\sigma(j)j}$ for all $\sigma \in S_n$, and define $\tau = (i, j) \in S_n$. Then:

# Existence (IV)

**Proof.**

$$
\begin{aligned}
& \det(C_i, \ldots, C_i, \ldots, C_j, \ldots, C_n) \\
= & \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{\sigma(1)1} \cdots a_{\sigma(i)i} \cdots a_{\sigma(j)j} \cdots a_{\sigma(n)n} \\
= & \sum_{\sigma \in A_n} a_{\sigma(1)1} \cdots a_{\sigma(i)i} \cdots a_{\sigma(j)j} \cdots a_{\sigma(n)n} \\
- & \sum_{\sigma \in A_n} a_{\sigma\tau(1)1} \cdots a_{\sigma\tau(i)i} \cdots a_{\sigma\tau(j)j} \cdots a_{\sigma\tau(n)n} \\
= & \sum_{\sigma \in A_n} a_{\sigma(1)1} \cdots a_{\sigma(i)i} \cdots a_{\sigma(j)j} \cdots a_{\sigma(n)n} \\
- & \sum_{\sigma \in A_n} a_{\sigma(1)1} \cdots a_{\sigma(j)i} \cdots a_{\sigma(i)j} \cdots a_{\sigma(n)n} = 0.
\end{aligned}
$$

# Existence (and V)

**Proof.**

Finally, we have $1_n = (\delta_{ij})$ where $\delta_{ij}$ is the Kronecker delta. Thus:

$$\det(1_n) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)\delta_{\sigma(1)1} \cdots \delta_{\sigma(n)n} = \delta_{11} \cdots \delta_{nn} = 1.$$

Hence det is indeed a determinant. □

# The determinant as a natural transformation

Consider **AbRing** the category of commutative rings, **Grp** the category of groups.

For each $n \in \mathbb{N}$, the general linear group $GL_n(-)$ is a functor from **AbRing** to **Grp**. Moreover the operation $(-)^\times$ sending an abelian ring to its group of units is also a functor from **AbRing** to **Grp**.

The determinant det is a natural transformation $\det : GL_n(-) \longrightarrow (-)^\times$.

# More elaborated determinants

- Given $R$ a commutative ring with unit, we can define a determinant for an endomorphism $T$ of a free $R$ module $M$ of rank $n$:

$$T(m_1) \wedge \cdots \wedge T(m_n) = \det(T) \cdot (m_1 \wedge \cdots \wedge m_n).$$

- There are determinants of complexes and categories of determinants.

# Something to take home

- Basics concepts in Mathematics are extremely powerful. Never underestimate how useful they can be, even to tackle problems that seem out of their reach.
- Linear Algebra appears absolutely everywhere, and a deep understanding of it will provide insight into more complex concepts.

*Thank you!*

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