Hochschild Cohomology of Twisted Tensor Product Algebras

Tekin Karadag, Dustin McPhate, Pablo S. Ocal, Tolulope Oke, Sarah Witherspoon

Texas A&M University

Motivation

Consider A and B two algebras over k a field. It was proven by Le and Zhou in [3] that, when considered with the correct structures, we have:

$$HH^*(A \otimes B) \cong HH^*(A) \otimes HH^*(B)$$

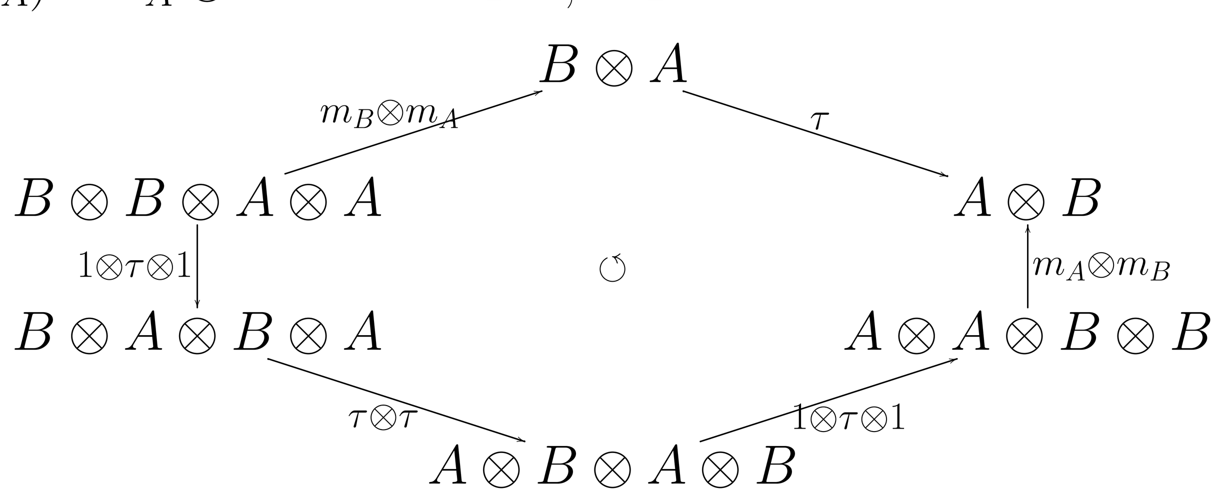
as Gerstenhaber algebras. Notice that this involves both understanding the natural Gerstenhaber algebra structure on $HH^*(A \otimes B)$ as well as providing a suitable structure to $HH^*(A) \otimes HH^*(B)$.

Goal

We want to understand $HH^*(A \otimes_\tau B)$ in terms of $HH^*(A)$ and $HH^*(B)$. More specifically, given a resolution of A as A^e module, and a resolution of B as B^e module, we compute a resolution of $A \otimes_\tau B$ as $(A \otimes_\tau B)^e$ module.

Setup [1] [2] [4]

Let A, B two algebras over k. We say that a bijective k linear map $\tau : B \otimes A \longrightarrow A \otimes B$ is a twisting map if $\tau(1_B \otimes a) = a \otimes 1_B$ and $\tau(b \otimes 1_A) = 1_A \otimes b$ for all $a \in A$, $b \in B$ and:



under this condition, the twisted tensor product algebra $A \otimes_\tau B$ is the vector space $A \otimes B$ with multiplication:

$$m_\tau : (A \otimes B) \otimes (A \otimes B) \xrightarrow{1 \otimes \tau \otimes 1} A \otimes A \otimes B \otimes B \xrightarrow{m_A \otimes m_B} A \otimes B.$$

We say that an A bimodule M, whose bimodule structure is given by $\rho_A : A \otimes M \otimes A \longrightarrow M$, is compatible with τ if there exist a bijective k linear map $\tau_{B,M} : B \otimes M \longrightarrow M \otimes B$ such that:

❶ $\tau_{B,M}$ is well behaved with respect to the algebra structure of B,

❷ $\tau_{B,M}$ is well behaved with respect to ρ_A the module structure of M via the twisting map τ.

Analogously, a B bimodule N can be compatible with τ via $\tau_{N,A}$. Given M and N bimodules over A and B via ρ_A and ρ_B compatible with τ via $\tau_{B,M}$ and $\tau_{N,A}$ respectively, then $M \otimes N$ has a natural bimodule structure over $A \otimes_\tau B$ given by:

$$\rho_{A \otimes_\tau B} = (\rho_A \otimes \rho_B) \circ (1 \otimes 1 \otimes \tau \otimes 1 \otimes 1) \circ (1 \otimes \tau_{B,M} \otimes \tau_{N,A} \otimes 1).$$

Compatibility of resolutions

Given M an A bimodule that is compatible with τ, we say that a projective A^e resolution $P_\bullet(M)$ is compatible with τ if each $P_i(M)$ is compatible with τ via a map $\tau_{B,i} : B \otimes P_i(M) \longrightarrow P_i(M) \otimes B$ such that $\tau_{B,\bullet}$ is a chain map lifting $\tau_{B,M}$.

Given N a B bimodule compatible with τ, we can analogously define how a projective B^e resolution $P_\bullet(N)$ is compatible with τ via $\tau_{\bullet,A}$.

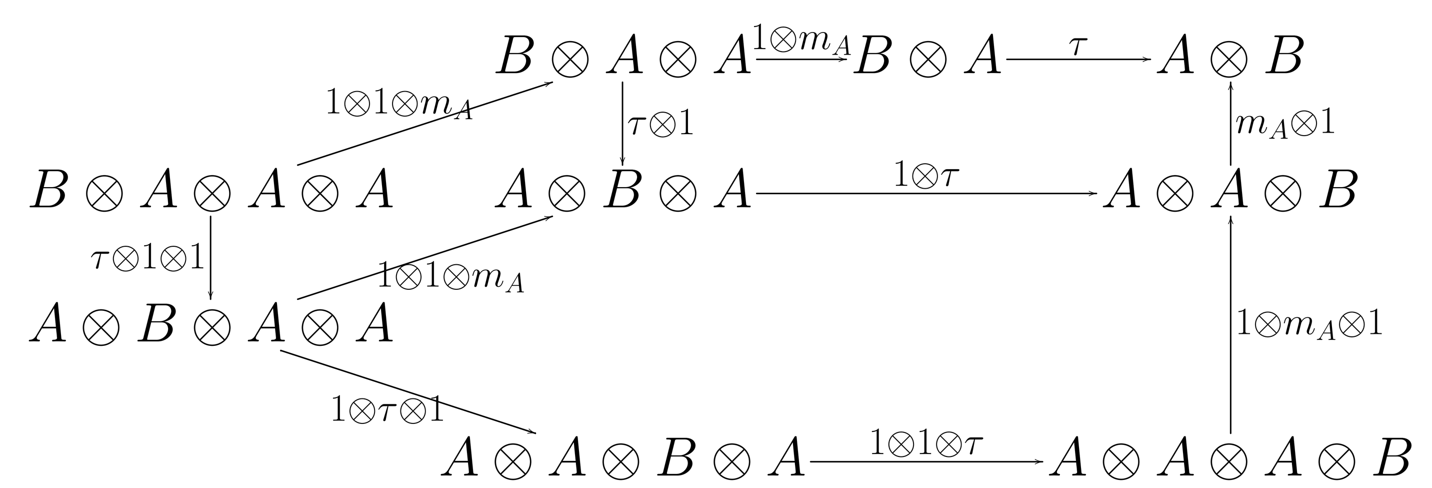
Proposition [4]

Let τ be a twisting map for A and B. Then $\mathbb{B}(A)$ and $\mathbb{B}(B)$, the bar resolutions of A and B respectively, are compatible with τ.

Technique(s)

We need to define the maps guaranteeing compatibility. For each $n \in \mathbb{N}$ define the maps $\tau_{B,n} : B \otimes \mathbb{B}_n(A) \longrightarrow \mathbb{B}_n(A) \otimes B$ recursively: $\tau_{B,0} := 1 \otimes \tau \circ \tau \otimes 1$, $\tau_{B,n} := 1 \otimes \tau \circ \tau_{B,n-1} \otimes 1$. We define analogously $\tau_{n,A}$.

For both A and B to satisfy the prerequisites of compatibility necessary to ask whether $\mathbb{B}(A)$ and $\mathbb{B}(B)$ may be compatible with τ, we focus on compatibility with the module structure:



the top right and bottom diagrams are commutative by compatibility with the algebra structure, and the left square is commutative because the functions are acting in terms of the tensor product that do not interfere with each other. We proceed analogously for B.

To see that $\mathbb{B}(A)$ is compatible with τ we need that for all $n \in \mathbb{N}$:

❶ Commutativity with the product in B:

$$\tau_{B,n} \circ m_B \otimes 1 = 1 \otimes m_B \circ \tau_{B,n} \otimes 1 \circ 1 \otimes \tau_{B,n}.$$

❷ Commutativity with the bimodule structure:

$$\tau_{B,n} \circ 1 \otimes \rho_{A,n} = \rho_{A,n} \otimes 1 \circ 1 \otimes 1 \otimes \tau \circ 1 \otimes \tau_{B,n} \otimes 1 \circ \tau \otimes 1 \otimes 1.$$

❸ Lifting to a chain map:

$$\tau_{B,n+1} \circ 1 \otimes d_n = d_n \otimes 1 \circ \tau_{B,n+2}.$$

The second part of the statement follows analogously.

Key idea(s)

We can see the Koszul resolution as lying entirely inside the bar resolution, and these techniques over diagrams still apply to it.

In fact, we can use these techniques to generalize a result (for graded algebras) in [2] that gives an explicit formula for the Gerstenhaber bracket of $HH^*(R \otimes^t S)$ in terms of elements in $HH^*(R)$ and $HH^*(S)$, as long as we work over the bar or the Koszul resolutions.

Proposition

Resolving A and B via the bar or the Koszul resolutions, we can explicitly compute the Gerstenhaber bracket of $HH^*(A \otimes_\tau B)$ in terms of elements in $HH^*(A)$ and $HH^*(B)$.

Applications [2] [5]

- Truncated algebras: for some $q \in k^*$, $k\langle x, y\rangle / (x^2, y^2, xy + qyx)$.
- Jordan plane: $k\langle x, y\rangle / (xy - yx - y^2)$.

References

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Contact Information

- **Web:** http://www.math.tamu.edu/~pso
- **Email:** pso@math.tamu.edu