Toward Free Resolutions Over Scrolls

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The Minimal Free Resolution

For a commutative (graded) local ring $(R, \mathfrak{m})$ and an R-module M, the *minimal free resolution* of M over R is the complex

$$\mathcal{F}_\bullet : \cdots \to F_i \xrightarrow{\partial_i} F_{i-1} \to \cdots \xrightarrow{\partial_3} F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \to M \to 0$$

where

- the sequence is exact, i.e. $\operatorname{Im}(\partial_k) = \ker(\partial_{k-1})$
- each of the F_k is a finitely generated free R-module, that is, $F_k \cong R^{\beta_k}$ for some β_k
- $\partial_k(F_k) \subseteq \mathfrak{m}F_{k-1}$ (essentially, this makes the β_k as small as possible)

We call the β_k the *Betti numbers* of M, sometimes written $\beta_k^R(M)$.

Important results over the polynomial ring $S = \Bbbk[x_1, \ldots, x_n]$ include:

Hilbert's Syzygy Theorem [Hilbert, 1890]

If M is a finitely generated module over S, then there exists a free resolution of M that has $F_i = 0$ for $i > n$.

Resolution Recipes

There are many recipes for *combinatorial* resolutions over S.

For a general ring R, we have no such luck.

- Failure of Hilbert's Syzygy Theorem means that our resolution might **not** terminate.
- Our resolution recipes don't usually work either.

In general, infinite free resolutions are not well understood.

QUESTION: For a special enough R, can we cobble something together?

Toric Rings

Write $\mathcal{A} = \{a_1, \ldots, a_n\} \subseteq \mathbb{N}^d$ (and sometimes for the $d \times n$ matrix with columns a_i), where $d \leq n$ and $\operatorname{rank} \mathcal{A} = d$. Define the map

$$\begin{aligned} \varphi : \Bbbk[x_1, \ldots, x_n] &\to \Bbbk[t_1, \ldots, t_d] \\ x_i &\mapsto \mathbf{t}^{a_i} = t_1^{a_{i,1}} \cdots t_d^{a_{i,d}} \end{aligned}$$

So for monomials $\mathbf{x}^u$, $\varphi(\mathbf{x}^u) = \mathbf{t}^{\mathcal{A}u}$.

Toric Ideal

The kernel $I_\mathcal{A}$ of φ is a prime binomial ideal called the *toric ideal* associated to $\mathcal{A}$. The ideal can be written $I_\mathcal{A} = \langle \mathbf{x}^u - \mathbf{x}^v \mid \mathcal{A}u = \mathcal{A}v \rangle$.

Toric Ring

The *toric ring* associated to $\mathcal{A}$ is

$$R = \Bbbk[x_1, \ldots, x_n]/I_\mathcal{A} \cong \Bbbk[\mathbf{t}^{a_1}, \mathbf{t}^{a_2}, \ldots, \mathbf{t}^{a_n}].$$

Example

The usual polynomial ring is a special example of a toric ring when $\mathcal{A} = \{e_1, e_2, \ldots, e_n\} \subseteq \mathbb{N}^n$.

Example

If $\mathcal{A} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\} \subseteq \mathbb{N}^2$, then $\left\{ \begin{array}{ccc} x_1 & \mapsto & t_1 \\ x_2 & \mapsto & t_1 t_2 \\ x_3 & \mapsto & t_1 t_2^2 \end{array} \right\}$ and

$$R = \Bbbk[x_1, x_2, x_3]/\langle x_1 x_3 - x_2^2 \rangle \cong \Bbbk[t_1, t_1 t_2, t_1 t_2^2].$$

Question: For a ~~special enough~~ toric ring R, can we cobble something together? (Where "cobble something together" means explicitly describe any/a minimal free resolution over R.)

The toric rings we will consider are those tied to *rational normal k-scrolls.*

Rational Normal k-Scroll

The *rational normal k-scroll* $\mathcal{S}(m_1 - 1, m_2 - 1, \ldots, m_k - 1)$ is the variety in $\mathbb{P}^{n-1}$ defined by the ideal $I_2(M)$ of 2×2 minors of the $2 \times (n-k)$ matrix $M =$

$$\left[\begin{array}{cccc|ccc|c|ccc} x_{1,1} & x_{1,2} & \ldots & x_{1,m_1-1} & x_{2,1} & \ldots & x_{2,m_2-1} & \ldots\ldots & x_{k,1} & \ldots & x_{k,m_k-1} \\ x_{1,2} & x_{1,3} & \ldots & x_{1,m_1} & x_{2,2} & \ldots & x_{2,m_2} & \ldots\ldots & x_{k,2} & \ldots & x_{k,m_k} \end{array}\right],$$

where $\sum_{i=1}^{k} m_i = n$.

One can show that $I_2(M)$ defining $\mathcal{S}(m_1 - 1, \dots, m_k - 1)$ is in fact the toric ideal associated to

$$\mathcal{A} = \left[\begin{array}{cccc|cccc|cc|cccc} 1 & \dots & \dots & 1 & 0 & \dots & \dots & 0 & \dots & \dots & 0 & \dots & \dots & 0 \\ 0 & \dots & \dots & 0 & 1 & \dots & \dots & 1 & 0 & \dots & 0 & \dots & \dots & 0 \\ \vdots & & & \vdots & & & & & & & \vdots & & & \vdots \\ 0 & \dots & \dots & 0 & 0 & \dots & \dots & 0 & \dots & 0 & 1 & \dots & \dots & 1 \\ 0 & 1 & \dots & m_1 - 1 & 0 & 1 & \dots & m_2 - 1 & \dots & \dots & 0 & 1 & \dots & m_k - 1 \end{array}\right],$$

so $R = S/I_2(M)$ has a wealth of combinatorial structure.

GOAL
Our goal from here on out is to resolve the ground field $\Bbbk$ over R.

Rational Normal Curves

When $k = 1$, we call $\mathcal{S}(n-1)$ the *rational normal curve*, a curve in $\mathbb{P}^{n-1}$. In this case,

$$M = \begin{bmatrix} x_1 & x_2 & \dots & x_{n-1} \\ x_2 & x_3 & \dots & x_n \end{bmatrix}.$$

Betti numbers of $\Bbbk$ [Peeva-Reiner-Sturmfels (1998), [2]]

The Betti numbers of $\Bbbk$ are

$$\beta_i^R(\Bbbk) = \begin{cases} 1 & \text{for } i = 0 \\ n & \text{for } i = 1 \\ (n-1)^2(n-2)^{i-2} & \text{for } i \geq 2 \end{cases}$$

Resolution of Monomial Ideals over Rational Normal Curves [Gasharov-Horwitz-Peeva (2008), [1]]

Gives (explicitly) the minimal free resolution over R of the field $\Bbbk$.

Betti Numbers of $\Bbbk$ [Matusevich-S (2019)]

Let $I_2(M)$ define the rational normal k-scroll $\mathcal{S}(m_1 - 1, \ldots, m_k - 1)$. If $R = S/I_2(M)$, then

$$\beta_i^R(\Bbbk) = \sum_{j=0}^{i} \binom{k+1}{j} (n-k-1)^{i-j}.$$

In particular, for $r \geq 1$, $\beta_{k+r}^R(\Bbbk) = (n-k)^{k+1}(n-k-1)^{r-1}$.

Resolution of $\Bbbk$ over 2-Scrolls [Matusevich-S (2019)]

Gives (explicitly) the minimal free resolution over R of the field $\Bbbk$.

Example for $\mathcal{S}(2,1)$

$$M = \left[\begin{array}{cc|c} x_1 & x_2 & x_4 \\ x_2 & x_3 & x_5 \end{array}\right] \text{ so } I_2(M) = \langle x_1x_3 - x_2^2, x_1x_5 - x_2x_4, x_2x_5 - x_3x_4 \rangle.$$

$$\partial_1 = \left| \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \right|$$

$$\partial_2 = \left| \begin{array}{ccccccccccccc}
x_2 & x_3 & x_5 & 0 & 0 & 0 & 0 & 0 & 0 & x_4 & 0 & 0 & 0 \\
-x_1 & -x_2 & -x_4 & x_2 & x_3 & x_5 & 0 & 0 & 0 & 0 & x_4 & 0 & 0 \\
0 & 0 & 0 & -x_1 & -x_2 & -x_4 & 0 & 0 & 0 & 0 & 0 & x_4 & x_5 \\
0 & 0 & 0 & 0 & 0 & 0 & x_2 & x_3 & x_5 & -x_1 & -x_2 & -x_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -x_1 & -x_2 & -x_4 & 0 & 0 & 0 & -x_3
\end{array} \right|$$

$$\partial_3 = \left| \begin{array}{ccccccccccccccccccccccccccc}
x_2 & x_3 & x_5 & x_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-x_1 & -x_2 & -x_4 & 0 & x_4 & x_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -x_1 & -x_2 & -x_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_4 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & x_2 & x_3 & x_5 & x_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -x_1 & -x_2 & -x_4 & 0 & x_4 & x_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x_1 & -x_2 & -x_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_2 & x_3 & x_5 & x_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x_3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x_1 & -x_2 & -x_4 & 0 & x_4 & x_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x_1 & -x_2 & -x_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x_3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x_2 & -x_3 & -x_5 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1 & x_2 & x_4 & -x_2 & -x_3 & -x_5 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1 & x_2 & x_4 & -x_2 & -x_3 & -x_5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1 & x_2 & x_4
\end{array} \right|$$

Thank you!

V. Gasharov, N. Horwitz, and I. Peeva.
Hilbert functions over toric rings.
Michigan Math. J., 57:339–357, 2008.
Special volume in honor of Melvin Hochster.

I. Peeva, V. Reiner, and B. Sturmfels.
How to shell a monoid.
Math. Ann., 310(2):379–393, 1998.

Mapping Cone for $\mathcal{S}(2,1)$

$$M = \left[\begin{array}{cc|c} x_1 & x_2 & x_4 \\ x_2 & x_3 & x_5 \end{array}\right] \text{ and } \mathcal{A} = \left[\begin{array}{ccc|cc} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 2 & 0 & 1 \end{array}\right]$$

Start with the short exact sequence:

$$0 \to \langle x_1, x_2, x_3 \rangle \cap \langle x_4, x_5 \rangle \to \langle x_1, x_2, x_3 \rangle \oplus \langle x_4, x_5 \rangle \to \langle x_1, \ldots, x_5 \rangle \to 0$$

Make resolutions of the first two, lift chain map to a resolution of the third.

R is a Koszul algebra

$\Rightarrow P_R(t) = 1/H(R; -t) = 1/H(S/\operatorname{in}_\prec I_\mathcal{A}; -t)$

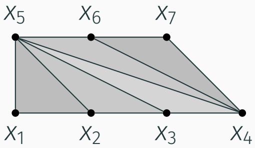

Figure 1: Stanley-Reisner complex for $\mathcal{S}(3,2)$

$$\mathcal{A} = \left[\begin{array}{cccc|ccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 0 & 1 & 2 \end{array}\right] \text{ and } M = \left[\begin{array}{ccc|cc} x_1 & x_2 & x_3 & x_5 & x_6 \\ x_2 & x_3 & x_4 & x_6 & x_7 \end{array}\right]$$

Here $\operatorname{in}_\prec I_\mathcal{A} = \langle x_1x_3, x_1x_4, x_1x_6, x_1x_7, x_2x_4, x_2x_6, x_2x_7, x_3x_6, x_3x_7, x_5x_7 \rangle$.

$$\partial_2 = \left[\begin{array}{cccc|ccccc}
\varphi_0 & & & & x_4 & & & & \\
& \varphi_0 & & & & x_4 & & & \\
& & & & & & x_4 & x_5 & x_6 \\ \hline
& & \varphi_0 & & -x_1 & -x_2 & -x_3 & & \\
& & & \varphi_0 & & & & -x_3 & \\
& & & & & & & & -x_3
\end{array}\right],$$

$$\partial_3 = \left[\begin{array}{c|cccc}
\varphi_1^{\oplus 4} & x_4 \cdot \mathbb{1}_8 & & & \\
& & & -x_3 \cdot \mathbb{1}_8 & \\ \hline
& -\varphi_0 & & & \\
& & -\varphi_0 & & \\
& & & -\varphi_0 & \\
& & & & -\varphi_0
\end{array}\right]$$

$$\partial_i = \left[\begin{array}{c|cc}
\varphi_{i-2}^{\oplus 4} & x_4 \cdot \mathbb{1}_{8\cdot 3^{i-3}} & \\
& & -x_3 \cdot \mathbb{1}_{8\cdot 3^{i-3}} \\ \hline
& \multicolumn{2}{c}{-\varphi_{i-3}^{\oplus 4}}
\end{array}\right]$$

Resolution for $\mathcal{S}(2,2)$

$$
\varphi_0 = \left[\begin{array}{cc|cc} x_2 & x_3 & x_5 & x_6 \\ -x_1 & -x_2 & -x_4 & -x_5 \end{array}\right]
$$

$$
\varphi_1 = \left[\begin{array}{cccc|cccc|cccc}
x_2 & x_3 & x_5 & x_6 & x_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-x_1 & -x_2 & -x_4 & -x_5 & 0 & x_4 & x_5 & x_6 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & -x_1 & -x_2 & -x_3 & 0 & x_2 & x_3 & x_5 & x_6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -x_3 & -x_1 & -x_2 & -x_4 & -x_5
\end{array}\right]
$$

$$
\varphi_2 = \left[\begin{array}{c|cccccccccccc|c}
\varphi_1 & x_1 & x_2 & x_4 & x_5 & & & & & & & & & \\
 & 0 & 0 & 0 & 0 & & & & & & & & & \\
\hline
 & -x_2 & -x_3 & -x_5 & -x_6 & & & & & & & & & \\
 & x_1 & x_2 & x_4 & x_5 & -x_2 & -x_3 & -x_5 & -x_6 & & & & & \\
 & & & & & x_1 & x_2 & x_4 & x_5 & -x_2 & -x_3 & -x_5 & -x_6 & \\
 & & & & & & & & & x_1 & x_2 & x_4 & x_5 & \\
\hline
 & & & & & & & & & 0 & 0 & 0 & 0 & \\
 & & & & & & & & & -x_2 & -x_3 & -x_5 & -x_6 & \varphi_1
\end{array}\right]
$$

$$
\varphi_i = \varphi_{i-1} \bigoplus \varphi_{i-2}^{\oplus 3} \bigoplus \varphi_{i-1}
$$