Max Intersection Complete Codes and the Factor Complex

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Outline

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- Background and Main Result
- The Factor Complex
- Three World Correspondence

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Notation:

- $[n]$ refers to $\{1, 2, \ldots, n\}$.
- For $\sigma \subseteq [n]$, let
$$\bar{\sigma} := \{\bar{i} \mid i \in \sigma\}$$
(e.g. $\overline{[n]} = \{\bar{1}, \bar{2}, \ldots, \bar{n}\}$).
- Correspondence between monomials and subsets of $[n] \cap \overline{[n]}$ as follows:
$$\prod_{i \in \sigma} x_i \prod_{j \in \tau} y_j \leftrightarrow \sigma \cup \bar{\tau}$$
(e.g. $x_1 x_2 x_3 y_2 y_4 \leftrightarrow \{1, 2, 3, \bar{2}, \bar{4}\}$).

Background

Definition

- A *neural code* $\mathcal{C}$ on n neurons is a collection of subsets of $[n]$.
- A collection $\mathcal{U} = \{U_i\}_{i=1}^n$ of open subsets of $\mathbb{R}^d$ is a *realization* of $\mathcal{C}$ if for all $\sigma \subseteq [n]$, we have

$$\bigcap_{i \in \sigma} U_i \setminus \bigcup_{j \notin \sigma} U_j \neq \emptyset \Leftrightarrow \sigma \in C.$$

- $\mathcal{C}$ is *convex* if there exists a realization $\mathcal{U}$ of $\mathcal{C}$ such that every $U_i \in \mathcal{U}$ is convex.

Background

Which neural codes are convex?

Definition

A code $\mathcal{C}$ is *max intersection complete* if any arbitrary intersection of maximal codewords of $\mathcal{C}$ is also in $\mathcal{C}$.

Theorem (Cruz, Giusti, Itskov, Kronholm, 2017)

Max intersection complete codes are convex.

Question (Curto, Gross, Jeffries, Morrison, Rosen, S, Youngs, 2018)

Is there an algebraic signature for max intersection completeness in the neural ideal?

Main Result

Definition

The *neural ideal* $J_{\mathcal{C}}$ of $\mathcal{C}$ is

$$J_{\mathcal{C}} := \langle \{ \prod_{i \in c} x_i \prod_{j \notin c} (1 - x_j) \mid c \in 2^{[n]} \setminus \mathcal{C} \} \rangle$$

Theorem (RdP, S, M)

$\mathcal{C}$ is max intersection complete iff for every minimal pseudomonomial ϕ in $J_{\mathcal{C}}$, with ϕ not a monomial, there exists $i \in [n]$ such that

(i) $(1 - x_i) \mid \phi$, *and*

(ii) *every minimal prime of $I(\Delta(\mathcal{C}))$ that contains x_i also contains ϕ.*

Polarization

Definition (Gunturkun, Jeffries, Sun, 2017)

The *polarization* of the pseudomonomial

$$\phi = \prod_{i \in \sigma} x_i \prod_{j \in \tau} (1 - x_j)$$

is

$$\mathcal{P}(\phi) := \prod_{i \in \sigma} x_i \prod_{j \in \tau} y_j .$$

The *polarization* of the pseudomonomial ideal

$$J = \langle \phi_1, \phi_2 \ldots, \phi_k \rangle$$

is

$$\mathcal{P}(J) := \langle \mathcal{P}(\phi_1), \mathcal{P}(\phi_2), \ldots, \mathcal{P}(\phi_k) \rangle .$$

The Factor Complex

To get the *factor complex* $\Delta_\cap(\mathcal{C})$ of a code $\mathcal{C}$:

- Take the *neural ideal* $J_\mathcal{C}$
- Take the *minimal primes* P_1, P_2,...,P_l of $J_\mathcal{C}$
- Polarize each minimal prime, and then consider the ideal

$$\mathcal{P}_\cap(J_\mathcal{C}) := \bigcap_{t=1}^{l} \mathcal{P}(P_t).$$

- Then $\Delta_\cap(\mathcal{C})$ is the simplicial complex for which $\mathcal{P}_\cap(J_\mathcal{C})$ is the Stanley-Reisner ideal. That is,

$$\Delta_\cap(\mathcal{C}) := \{\sigma \cup \overline{\tau} \mid \prod_{i\in\sigma} x_i \prod_{j\in\tau} y_j \notin \mathcal{P}_\cap(J_\mathcal{C})\}.$$

More Definitions

Definition

- The *complement code* of $\mathcal{C}$ on n neurons is $\mathcal{C}' := 2^{[n]} \smallsetminus \mathcal{C}$.
- $c, d \subseteq [n]$ with $c \subseteq d$. Their *interval* is $[c, d] := \{w \subseteq [n] \mid c \subseteq w \subseteq d\}$
- Δ a simplicial complex on $[n] \cup \overline{[n]}$. $B \subseteq \{\bar{1}, \bar{2}, \ldots, \bar{n}\}$ is a *prime set* of Δ if $F \in \Delta,\ F \supseteq [n] \Rightarrow B \not\subseteq F$.

Example: Let Δ be a simplicial complex on $\{1, 2, 3, 4, 5, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$ with facets

- $\{1, 2, 3, 4, 5, \bar{1}, \bar{2}, \bar{3}\}$,
- $\{1, 2, 3, 4, 5, \bar{2}, \bar{3}, \bar{5}\}$, and
- $\{1, 2, 3, 4, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$.

Then the minimal prime sets of Δ are $\{\bar{4}\}$ and $\{\bar{1}, \bar{5}\}$.

Threefold Correspondence, Part I

Information about what codewords are contained in $\mathcal{C}$ lies in each of the following:

(1) intervals of $\mathcal{C}$,

(2) effective faces of $\Delta_\cap(\mathcal{C})$, and

(3) pseudomonomials of $J_{\mathcal{C}'}$.

Theorem (Curto, Itskov, Veliz-Cuba, Youngs, 2013; Gunturkun, Jeffries, Sun, 2017; RdP, S, M)

The following are equivalent:

(1) $[c, d] \subseteq \mathcal{C}$

(2) $d \cup \overline{[n] \setminus c} \in \Delta_\cap(\mathcal{C})$

(3) $\prod_{i \in c} x_i \prod_{j \in [n] \setminus d} (1 - x_j) \in J_{\mathcal{C}'}$.

Threefold Correspondence, Part II

Information about what codewords are maximal lies in each of the following:

(1) max codewords of $\mathcal{C}$,

(2) minimal prime sets of $\Delta_\cap(\mathcal{C}')$, and

(3) minimal primes of $I(\Delta(\mathcal{C}))$.

Theorem (RdP, S, M)

The following are equivalent:

(1) *c maximal in $\mathcal{C}$*

(2) *$\overline{[n] \smallsetminus c}$ is a minimal prime set of $\Delta_\cap(\mathcal{C}')$.*

(3) *$\langle \{x_i \mid i \in [n] \smallsetminus c\} \rangle$ is a minimal prime of $J_{\mathcal{C}}$.*

Max $\cap$-Completeness in the Factor Complex

Theorem (RdP, S, M)

$\mathcal{C}$ is max intersection complete iff for every facet F of $\Delta_\cap(\mathcal{C}')$ that does not contain $[n]$, there exists $i \in [n]$ such that

(i) *$i \notin F$, and*

(ii) *every minimal prime set of $\Delta_\cap(C')$ that contains $\bar{i}$ also contains some $\bar{j}$ such that $\bar{j} \notin F$.*

Theorem (Main Result)

$\mathcal{C}$ is max intersection complete iff for every minimal pseudomonomial ϕ in $J_\mathcal{C}$, with ϕ not a monomial, there exists $i \in [n]$ such that

(i) *$(1 - x_i) \mid \phi$, and*

(ii) *every minimal prime of $I(\Delta(\mathcal{C}))$ that contains x_i also contains ϕ.*

Thank You

Thanks for listening!