Resolutions for truncated Ore extensions

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Twisted Tensor Products

Let $\Bbbk$ be a field and A, B unital associative $\Bbbk$ algebras with multiplication maps m_A and m_B.

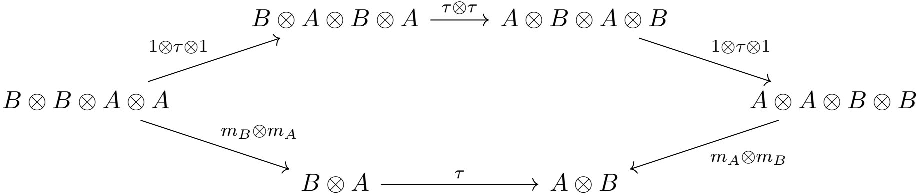
Definition (Twisting Map)

A **twisting map**, τ is a bijective $\Bbbk$-linear map

$$\tau : B \otimes A \to A \otimes B$$

for which $\tau(1_B \otimes a) = a \otimes 1_B$, $\tau(b \otimes 1_A) = 1_A \otimes b$, and

$$\tau \circ (m_B \otimes m_A) = (m_A \otimes m_B) \circ (1 \otimes \tau \otimes 1) \circ (\tau \otimes \tau) \circ (1 \otimes \tau \otimes 1)$$



Twisted Tensor Products

Definition (Twisted Tensor Product)

The **twisted tensor product algebra** $A \otimes_\tau B$ is the vector space $A \otimes_{\Bbbk} B$ with multiplication given by the map $(m_A \otimes m_B) \circ (1 \otimes \tau \otimes 1)$ on $A \otimes B \otimes A \otimes B$.

Example (Quantum Plane)

$$\Bbbk\langle x, y\rangle/(xy - qyx)$$

where $q \in \Bbbk$ and $q \neq 0$. Letting $A = \Bbbk\langle x\rangle$ and $B = \Bbbk\langle y\rangle$ with

$$\tau(y \otimes x) = q^{-1} x \otimes y$$

then $\Bbbk\langle x, y\rangle/(xy - qyx) \cong A \otimes_\tau B$

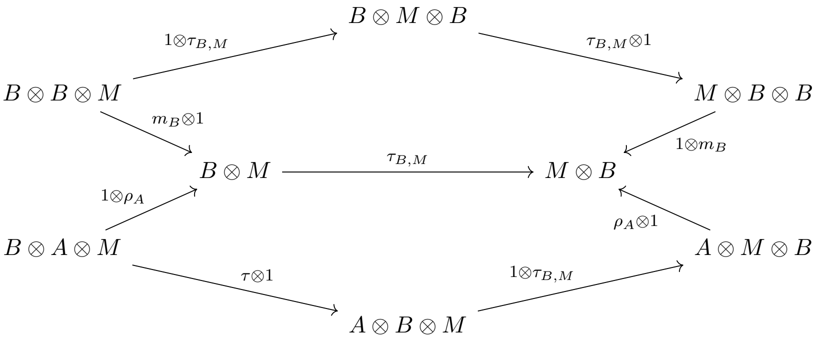
Modules

Definition (Compatability)

A left A-module M is said to be **compatible with** τ if $\exists$ a bijective $\mathbb{k}$-linear map

$$\tau_{B,M} : B \otimes M \to M \otimes B$$

which commutes with the module structure of M and multiplication in B



Resolutions

Let M be a left A-module compatible with a twisting map τ via some $\tau_{B,M}$. Let $P_\bullet(M)$ be a projective resolution of M as an A-module.

Definition

The resolution $P_\bullet(M)$ is said to be **compatible with** τ if each $P_i(M)$ is compatible with τ via a bijective $\Bbbk$-linear map

$$\tau_{B,i} : B \otimes P_i(M) \to P_i(M) \otimes B$$

with $\tau_{B,\bullet}$ lifting $\tau_{B,M}$.

Ore Extensions

Let A be a unital associative $\mathbb{k}$-algebra, $\sigma \in \mathrm{Aut}_{\mathbb{k}}(A)$, and δ be a σ-derivation. That is $\delta(aa') = \sigma(a)\delta(a') + \delta(a)a'$.

Definition (Ore extension)

The **Ore extension** $A[x;\sigma,\delta]$ is the associative algebra with underlying vector space $A[x]$ and multiplication determined by that of A an $\mathbb{k}[x]$ with the additional Ore relation

$$xa = \sigma(a)x + \delta(a)$$

Example (Quantum Plane)

$$\mathbb{k}\langle x, y\rangle/(xy - qyx)$$

where $q \in \mathbb{k}$ and $q \neq 0$. Letting $A = \mathbb{k}[x]$, $\sigma(x) = q^{-1}x$ and $\delta = 0$ then

$$\mathbb{k}\langle x, y\rangle/(xy - qyx) \cong A[y;\sigma,\delta]$$

More Examples

Example (first Weyl Algebra)

The first Weyl Algebra $\mathcal{W}$ is defined as

$$\mathcal{W} := \Bbbk\langle x, y\rangle/(xy - yx - 1)$$

Letting $A = \Bbbk[x]$, $\sigma = id_A$ and δ be formal differentiation of polynomials. Then $\mathcal{W} \cong A[y; \sigma, \delta]$

Example (Universal Enveloping Algebras)

Let $\mathfrak{g}$ be a Lie algebra . The universal enveloping algebra of $\mathfrak{g}$ is defined as the algebra with underlying vector space $\mathfrak{g}$ and multiplication defined by the Ore relation on generators

$$uv = vu + [u, v]$$

Truncated Ore Extensions

Let A be an associative $\Bbbk$-algebra, $\sigma \in \mathrm{Aut}_{\Bbbk}(A)$, and δ be a σ-derivation.

Definition

The **truncated Ore extension** $A[\overline{x}; \sigma, \delta]$, is the associative algebra with underlying vector space $A[x]/(x^n)$ and multiplication determined by that of A and $\Bbbk[x]/(x^n)$ with the additional Ore relation

$$\overline{x}a = \sigma(a)\overline{x} + \delta(a)$$

Example (Nichols Algebra)

$$\mathfrak{B}(V_0) = \Bbbk[x, y]/(x^2, y^2, xy + yx)$$

Letting $A = \Bbbk[x]/(x^2)$, $\sigma(x) = -x$, and $\delta = 0$ then for $n = 2$

$$\mathfrak{B}(V_0) \cong A[\overline{y}; \sigma, \delta].$$

Multiplication in Truncated Ore Extensions

We introduce some notation. Let $s_{(i_1,i_2,\ldots,i_k)}(x_1,x_2,\ldots,x_k)$ be the polynomial in k noncommuting variables which is the sum of all possible products of i_1 copies of x_1, i_2 copies of x_2, ..., and i_k copies of x_k

Example

$s_{(1,2)}(x,y) = xy^2 + yxy + y^2x$

Proposition

Let τ be a twisting map for the Ore extension $A[x;\sigma,\delta]$. If σ and δ satisfy the following conditions

$$s_{(i,j)}(\sigma,\delta) = 0$$

for $i+j=n$, $0 \leq i \leq n-1$, $1 \leq j \leq n$ then τ induces a well defined multiplication on the quotient $A[\overline{x};\sigma,\delta]$.

Compatibility with τ

Let A be any associative algebra and $B = \mathbb{k}[x]/(x^n)$. Suppose τ is a twisting map for $A[x;\sigma,\delta]$ and induces a well defined multiplication on $A \otimes_\tau B \cong A[\overline{x};\sigma,\delta]$.

Let M be a left $A \otimes_\tau B$-module where upon restriction to an A-module $\exists$ an A-module isomorphism

$$\phi : M \to M^\sigma$$

where M^σ is the vector space M with A-module action given by $a \cdot_\sigma m = \sigma(a) \cdot m$.

Definition

Let $\tau_{B,M} : B \otimes M \to M \otimes B$ be the $\mathbb{k}$-linear map induced by

$$\tau_{B,M}(\overline{x} \otimes m) = \phi(m) \otimes \overline{x} + \overline{x} \cdot m \otimes 1$$

Compatibility with τ

Let τ, $A[\overline{x};\sigma,\delta]$, M, and $\tau_{B,M}$ be defined as in the previous slide.

Lemma

If the maps ϕ and $\overline{x}\cdot$ satisfy the following relations

$$s_{(i,j)}(\phi,\overline{x}\cdot) = 0$$

for $i+j=n$ with $1 \leq i \leq n-1$, $1 \leq j \leq n-1$ then M is compatible with τ via $\tau_{B,M}$

Constructing $\tau_{B,\bullet}$

Let M be a left $A[\overline{x};\sigma,\delta]$-module compatible with τ via $\tau_{B,M}$ and $P_\bullet(M)$ be a projective resolution of M as an A-module. We then define $P_\bullet(M)^\sigma$ to be the vector spaces $P_i(M)$ with module action given by $a \cdot_\sigma z = \sigma(a) \cdot z$ then set $d_i^\sigma = d_i$ for $i \neq 0$ and $d_0^\sigma = \phi^{-1} d_0$.

Remark

By the comparison theorem $\exists$ an A-module chain map

$$\sigma_\bullet : P_\bullet(M) \to P_\bullet(M)^\sigma$$

lifting the identity on M.

Constructing $\tau_{B,\bullet}$

Lemma

For any projective A-module, P, $\exists$ an $A[\overline{x};\sigma,\delta]$-module structure on P that extends the action of A

Lemma

There exists a $\Bbbk$-linear chain map

$$\delta_\bullet : P_\bullet(M) \to P_\bullet(M)$$

which lifts the action of $\overline{x}$ on M such that for every $i \geq 0$, $a \in A$, $z \in P_i(M)$

$$\delta_i(a \cdot z) = \sigma(a)\delta_i(z) + \delta(a)z$$

Constructing $\tau_{B,\bullet}$

Definition

Let $\tau_{B,\bullet} : B \otimes P_\bullet(M) \to P_\bullet(M) \otimes B$ be the $\Bbbk$-linear chain map induced by

$$\tau_{B,i}(\overline{x} \otimes z) = \sigma_i(z) \otimes \overline{x} + \delta_i(z) \otimes 1$$

for all $z \in P_i(M)$.

Lemma

Let $\sigma_\bullet$ and $\delta_\bullet$ be the chain maps previously constructed. If $\sigma_\bullet$ and $\delta_\bullet$ satisfy the relations

$$s_{(i,j)}(\sigma_\bullet, \delta_\bullet) = 0$$

for $i + j = n$ with $0 \leq i \leq n - l$ and $1 \leq j \leq n$ then the resolution $P_\bullet(M)$ is compatible with the twisting map τ via $\tau_{B,\bullet}$.

Constructing Resolutions

Let A be any associative algebra, $B = \Bbbk[x]/(x^n)$, and $P_\bullet(B)$ be the standard projective resolution of $\Bbbk$ as a module over B with augmentaion map $\epsilon_B(\overline{x}) = 0$, i.e.

$$\cdots \xrightarrow{\overline{x}\cdot} B \xrightarrow{\overline{x}^{n-1}\cdot} B \xrightarrow{\overline{x}\cdot} B \xrightarrow{\epsilon_B} \Bbbk \longrightarrow 0$$

Let $A[\overline{x};\sigma,\delta]$ be a truncated Ore extension and M a left $A[\overline{x};\sigma,\delta]$-module for which $M \cong M^\sigma$ as A-modules and which is compatible with τ via $\tau_{B,M}$. Let $P_\bullet(M)$ be a projective resolution of M as an A-module which is compatible with τ via $\tau_{B,\bullet}$.

Theorem

If $\sigma_i : P_i(M) \to P_i(M)$ is bijective for every $i \geq 0$ then the twisted product complex of $P_\bullet(M)$ and $P_\bullet(B)$ gives a projective resolution of M as a left $A[\overline{x};\sigma,\delta]$-module.

Example

Let $\Bbbk$ be a field of prime characteristic p, $A = \Bbbk[x_1]/(x_1^p)$, and $B = \Bbbk[x_2]/(x_2^p)$. We consider the class of truncated Ore extensions of the form $A[\overline{x_2}; \sigma, \delta] \cong A \otimes_\tau B$ where

$$\tau(\overline{x_2} \otimes \overline{x_1}) = \sigma(\overline{x_1}) \otimes \overline{x_2} + \delta(\overline{x_1}) \otimes 1$$

with

$$\sigma = id_A$$

and δ is the σ-derivation defined by

$$\delta(1) = 0 \text{ and } \delta(\overline{x_1}) = \alpha\overline{x_1}^t$$

for $\alpha \in \Bbbk$ and $2 \leq t \leq p - 1$

Example

Remark

Since $\sigma = id_A$ then

$$s_{(i,j)}(\sigma, \delta) = \binom{p}{j} \delta^j$$

And since p is prime, $\mathrm{char}(\Bbbk) = p$, and $\delta^p(\overline{x_1}) = 0$ we have that

$$s_{(i,j)}(\sigma, \delta) = 0$$

Remark

Also we note that for any $m \in \Bbbk$ we have that $\sigma(a) \cdot m = a \cdot m$ and thus $\Bbbk$ is trivially isomorphic to $\Bbbk^\sigma$

Example

Definition

Letting $\phi = id_{\Bbbk}$ and noting that $\overline{x_1}$ acts on $\Bbbk$ as 0 we have

$$\tau_{B,\Bbbk}(b \otimes m) = m \otimes b$$

for all $b \in B$ and $m \in \Bbbk$

Let $P_\bullet(A)$ be the standard projective resolution of $\Bbbk$ as an A-module.

Proposition

$P_\bullet(A)$ is compatible with τ via the maps

$$\tau_{B,i}(\overline{x_2}^r \otimes \overline{x_1}^s) = \begin{cases} \tau(\overline{x_2}^r \otimes \overline{x_1}^s) = \sum_{j=0}^{r} \binom{r}{j} (s)^{[j]} (\alpha \overline{x_1}^t)^j \overline{x_1}^{s-j} \otimes \overline{x_2}^{r-j} \\ \sum_{j=0}^{r} \binom{r}{j} (s+1)^{[j]} (\alpha \overline{x_1}^t)^j \overline{x_1}^{s-j} \otimes \overline{x_2}^{r-j} \end{cases}$$

where $(s)^{[j]} = \prod_{i=0}^{j-1}(s + i(t-1)),\ (s)^{[0]} = 1$

Example

Let $P_\bullet(B)$ be the standard projective resolution of $\Bbbk$ as a B-module.

Proposition

$P_i(A) \otimes P_i(B)$ is a projective $A[\overline{x_2}; \sigma, \delta]$-module and thus the following twisted product complex is a projective resolution of $\Bbbk$ as a $A[\overline{x_2}; \sigma, \delta]$-module.

$$\cdots \xrightarrow{d_3} (A \otimes B)^{\oplus 3} \xrightarrow{d_2} (A \otimes B)^{\oplus 2} \xrightarrow{d_1} A \otimes B \longrightarrow \Bbbk \longrightarrow 0$$

with $d_k = \sum_{i+j=k} d_{i,j}$ for $d_{i,j} = (d_i \otimes 1) + ((-1)^i \otimes d_j)$

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