

Explicit Pieri Inclusions

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Integer partitions

Definition

A **partition** of $n \in \mathbb{N}$ is a sequence of integers

$$\lambda = (\lambda_1, \lambda_2, \dots)$$

where $\lambda_1 \geq \lambda_2 \geq \dots \geq 0$ and $\sum \lambda_i = n$.

- Only finitely many $\lambda_i \neq 0$
- Each $\lambda_i \neq 0$ is a *part* of λ
- $\lambda \vdash n$ or $|\lambda| = n$

$$\lambda = (4, 2, 1, 0, 0, \dots) = (4, 2, 1, 0, 0) = (4, 2, 1) \quad |\lambda| = 7$$

Young diagrams

Partitions of n can be represented by a *Young diagram* of size n , an array of n left-justified boxes with weakly decreasing row length.

Examples:

$$\lambda = (4, 2, 1) \quad \longleftrightarrow \quad \lambda = \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & & \\ \hline \square & & & \\ \hline \end{array}$$

$$\sigma = (6, 6, 3, 3, 1, 1) \quad \longleftrightarrow \quad \sigma = \begin{array}{|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square \\ \hline \square & \square & \square & \square & \square & \square \\ \hline \square & \square & \square & & & \\ \hline \square & \square & \square & & & \\ \hline \square & & & & & \\ \hline \square & & & & & \\ \hline \end{array}$$

Blocks of a diagram

Block notation: $\lambda = (w_1^{h_1}, w_2^{h_2} \dots, w_{N-1}^{h_{N-1}}, w_N^{h_N})$ where $w_i < w_{i+1}$ and each w_i appears as a part of λ exactly h_i times.

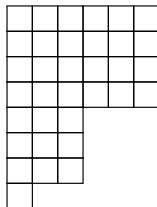
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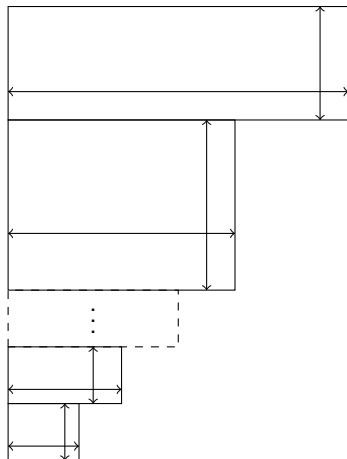


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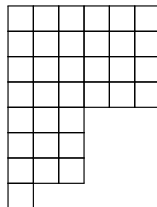
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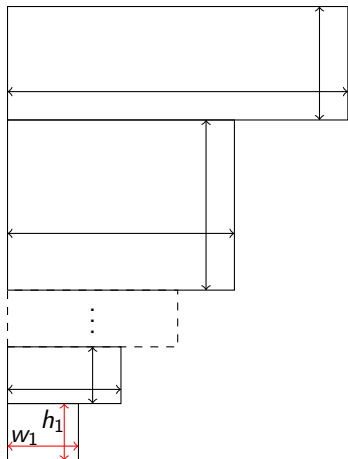


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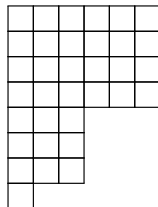
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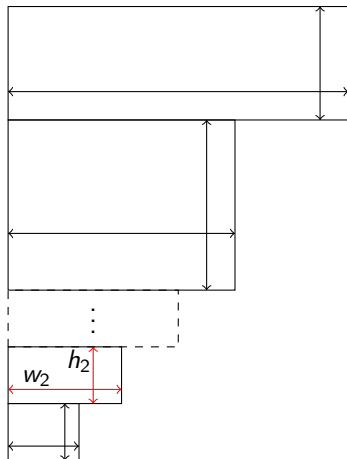


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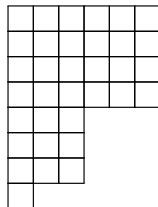
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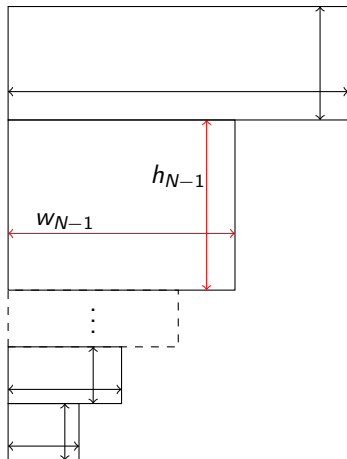


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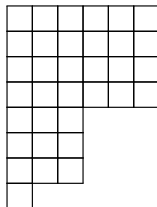
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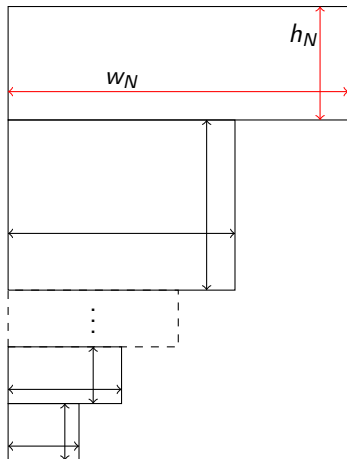


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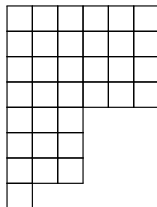
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Young tableaux

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4	2	9	1
7	5		
1			

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Semi-standard ones are weakly increasing across rows and strictly increasing down columns.

1	1	1	1	1	1
2	2	3	4	4	4
3	3	6			
4	5	7			
5					
8					

Representations of $GL_n(\mathbb{C})$

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Weyl modules are the reps of $GL_n(\mathbb{C})$ via this identification:

$$\lambda \longleftrightarrow \mathbb{S}_\lambda.$$

If $|\lambda| = m$,

- \mathbb{S}_λ can be constructed as subspaces (or quotients) of $(\mathbb{C}^n)^{\otimes m}$
- A basis is given by the semi-standard tableaux on λ with entries $1, \dots, n$

Weyl Modules

E.g.

$$S_{(m)} = \boxed{} \boxed{} \cdots \boxed{} \boxed{} = \text{Sym}^m(\mathbb{C}^n)$$

$$S_{(1, \dots, 1)} = \begin{array}{c} \boxed{} \\ \boxed{} \\ \vdots \\ \boxed{} \\ \boxed{} \end{array} = \bigwedge^m(\mathbb{C}^n)$$

The Pieri Rule

Theorem (Pieri Rule)

Let μ be a partition and $\nu = (1, \dots, 1)$ be a partition of m . Then we have an isomorphism of $GL_n(\mathbb{C})$ -modules

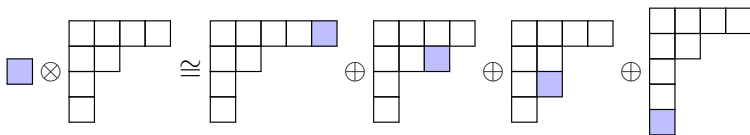
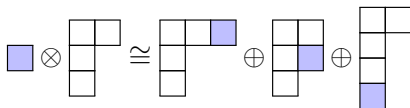
$$\mathbb{S}_\nu \otimes \mathbb{S}_\mu \cong \bigoplus_{\lambda} \mathbb{S}_\lambda$$

where the sum is over all $\lambda \supset \mu$ obtained by adding m boxes to μ with no two boxes in the same row. Similarly,

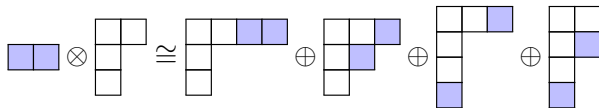
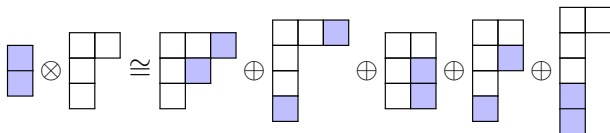
$$\mathbb{S}_{(m)} \otimes \mathbb{S}_\mu \cong \bigoplus_{\lambda} \mathbb{S}_\lambda$$

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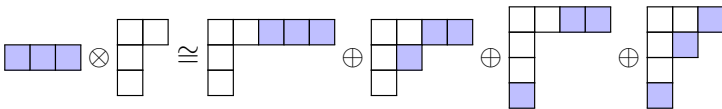
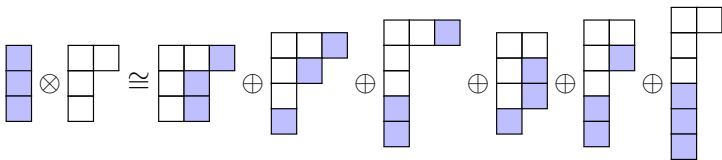
The Pieri Rule - One Box



The Pieri Rule - Two Boxes



The Pieri Rule - Three Boxes



Pieri Inclusions

From the Pieri rule we get maps

$$S_\lambda \rightarrow S_\nu \otimes S_\mu,$$

called **Pieri inclusions**, unique up to non-zero scalar multiple.

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The first explicit description of such inclusions was given by Olver (1982) in the “1-box removal” case

$$S_\lambda \xrightarrow{\Phi_0} S_{(1)} \otimes S_\mu$$

with the general case given by iteration.

Olver's description of Pieri inclusions

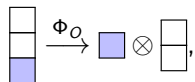
$$\Phi_O = \sum_J \frac{(-1)^{|J|} J}{c_J}$$

The J are the ways to remove the indicated box up and out of the diagram, $|J|$ is the number of rows used, and the c_J depend on the rows used.

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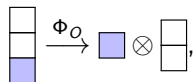
E.g., 

The diagram illustrates the operator Φ_O applied to a Young diagram. On the left is a vertical stack of three boxes, with the bottom box shaded blue. An arrow labeled Φ_O points to the right, where the result is shown as a tensor product: a single blue box followed by a circled \otimes symbol, and then a vertical stack of two boxes.

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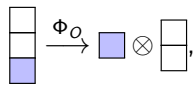
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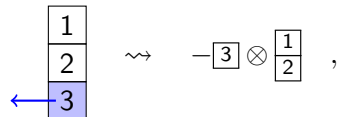
E.g.,  $\Phi_O \left(\begin{array}{c} \square \\ \square \\ \color{blue}\square \end{array} \right) = ?$

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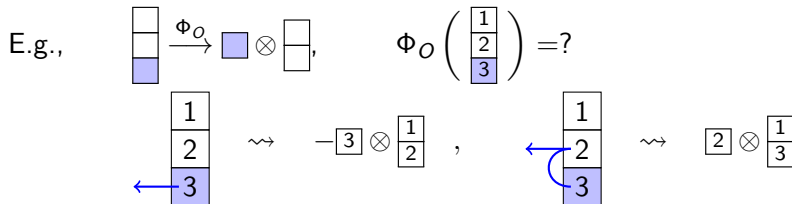
E.g.,  $\Phi_O \left(\begin{array}{|c|} \hline \square \\ \square \\ \color{blue}{\square} \\ \hline \end{array} \right) = \color{blue}{\square} \otimes \begin{array}{|c|} \hline \square \\ \square \\ \hline \end{array},$ $\Phi_O \left(\begin{array}{|c|} \hline 1 \\ 2 \\ \color{blue}{3} \\ \hline \end{array} \right) = ?$

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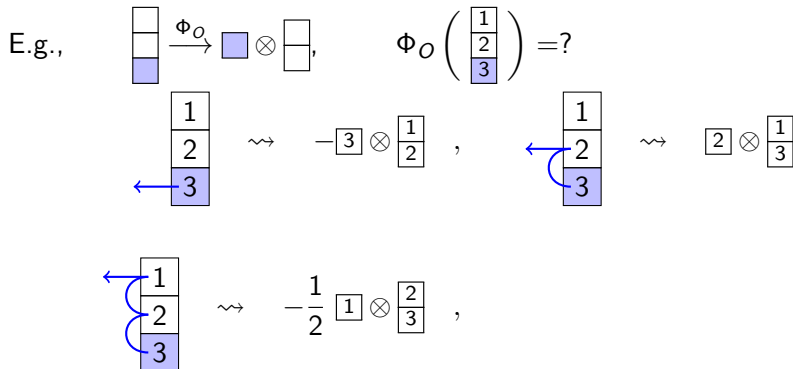
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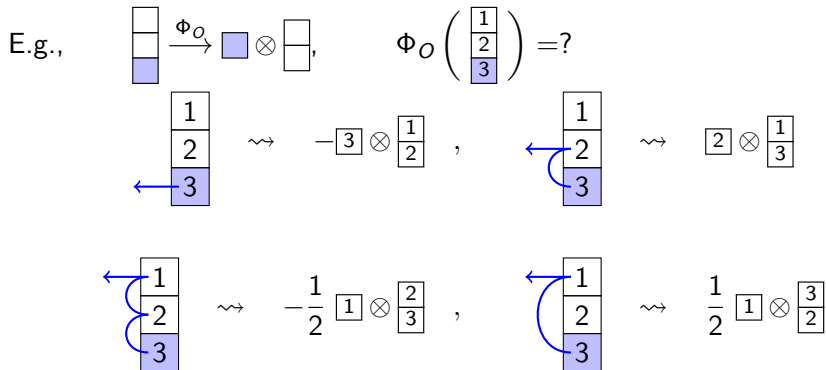
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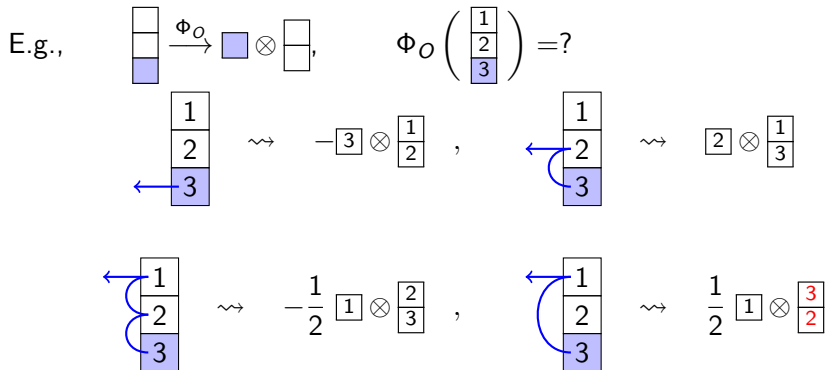
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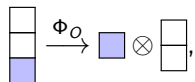
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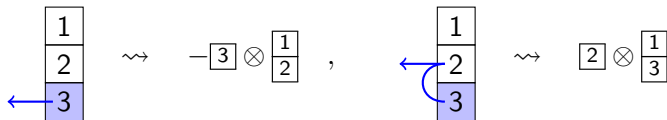
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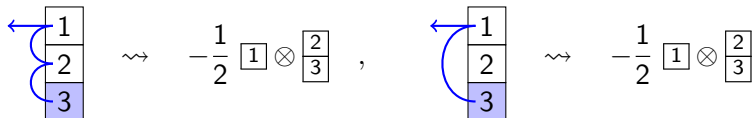
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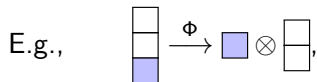
$$\Phi = \sum_P \frac{(-1)^{|P|} P}{H(P)}$$

The P are the ways to remove the indicated box up and out of the diagram *with a row skipping restriction*. The coefficients $H(P)$ are similar to the c_J , but depend only on the blocks used.

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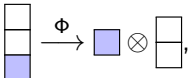
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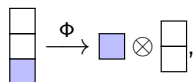
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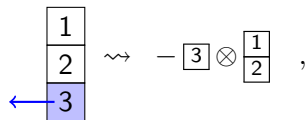
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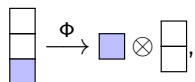
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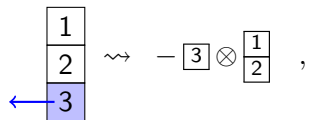
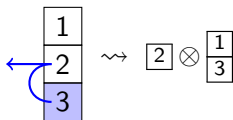
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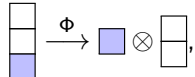
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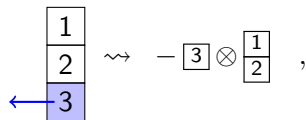
 $\sim - [3] \otimes \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array},$  $\sim [2] \otimes \begin{array}{|c|} \hline 1 \\ \hline 3 \\ \hline \end{array}$

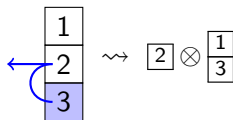
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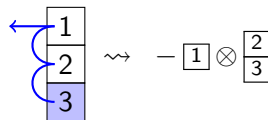
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E.g.,  $\Phi \left(\begin{array}{c} \square \\ \square \\ \blacksquare \end{array} \right) = ?$

 $\begin{array}{c} 1 \\ 2 \\ \blacksquare 3 \end{array} \rightsquigarrow - \square 3 \otimes \begin{array}{c} 1 \\ 2 \end{array},$

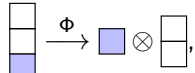
 $\begin{array}{c} 1 \\ 2 \\ \blacksquare 3 \end{array} \rightsquigarrow \square 2 \otimes \begin{array}{c} 1 \\ 3 \end{array}$

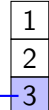
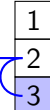

 $\begin{array}{c} 1 \\ 2 \\ \blacksquare 3 \end{array} \rightsquigarrow - \square 1 \otimes \begin{array}{c} 2 \\ 3 \end{array}$

New description of Pieri inclusions

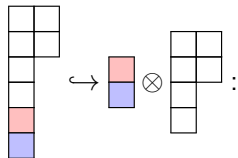
$$\Phi = \sum_P \frac{(-1)^{|P|} P}{H(P)}$$

The P are the ways to remove the indicated box up and out of the diagram *with a row skipping restriction*. The coefficients $H(P)$ are similar to the c_J , but depend only on the blocks used.

E.g.,  $\Phi \left(\begin{array}{c} \square \\ \square \\ \blacksquare \end{array} \right) = -\blacksquare \otimes \begin{array}{c} \square \\ \square \end{array} + \blacksquare \otimes \begin{array}{c} \square \\ \square \\ \square \end{array} - \blacksquare \otimes \begin{array}{c} \square \\ \square \\ \square \end{array}$

 $\rightsquigarrow -\square \otimes \begin{array}{c} \square \\ \square \\ \square \end{array},$  $\rightsquigarrow \square \otimes \begin{array}{c} \square \\ \square \\ \square \end{array}$  $\rightsquigarrow -\square \otimes \begin{array}{c} \square \\ \square \\ \square \end{array}$

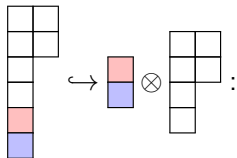
Pieri inclusions removing two boxes



A Young diagram consisting of a vertical column of 5 boxes, with the bottom two boxes colored red and blue. An arrow points to the following mathematical expression:

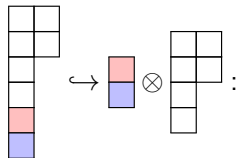
$$\sum_{\text{2-paths } P} \frac{(-1)^{|P|}}{H(P)} \begin{array}{|c|} \hline P_2 \\ \hline P_1 \\ \hline \end{array} \otimes P \left(\begin{array}{|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \\ \hline \square & \\ \hline \square & \\ \hline \end{array} \right)$$

Pieri inclusions removing two boxes

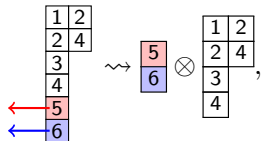


$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & 4 \\ \hline 3 & \\ \hline 4 & \\ \hline 5 & \\ \hline 6 & \\ \hline \end{array} \mapsto \sum_{\text{2-paths } P} \frac{(-1)^{|P|}}{H(P)} \begin{array}{|c|} \hline P_2 \\ \hline P_1 \\ \hline \end{array} \otimes P \left(\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline & \\ \hline & \\ \hline \end{array} \right)$$

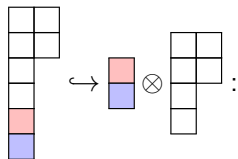
Pieri inclusions removing two boxes



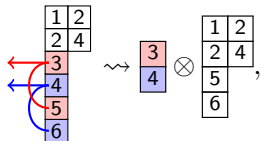
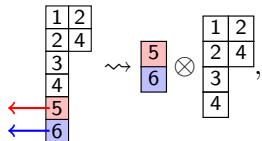
$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & 4 \\ \hline 3 & \\ \hline 4 & \\ \hline 5 & \\ \hline 6 & \\ \hline \end{array} \mapsto \sum_{\text{2-paths } P} \frac{(-1)^{|P|}}{H(P)} \begin{array}{|c|} \hline P_2 \\ \hline P_1 \\ \hline \end{array} \otimes P \left(\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline & \\ \hline & \\ \hline \end{array} \right)$$



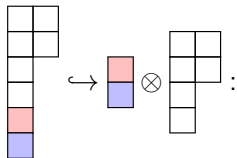
Pieri inclusions removing two boxes



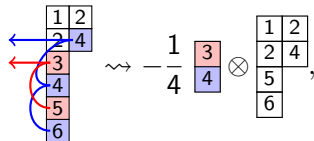
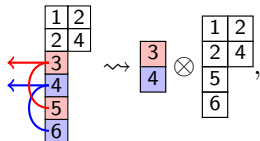
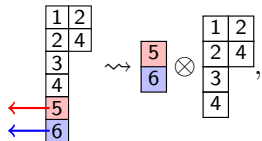
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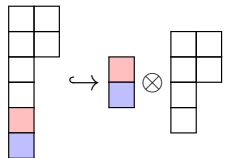
Pieri inclusions removing two boxes



$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & 4 \\ \hline 3 & \\ \hline 4 & \\ \hline 5 & \\ \hline 6 & \\ \hline \end{array} \mapsto \sum_{2\text{-paths } P} \frac{(-1)^{|P|}}{H(P)} \begin{array}{|c|} \hline P_2 \\ \hline P_1 \\ \hline \end{array} \otimes P \left(\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline & \\ \hline & \\ \hline & \\ \hline \end{array} \right)$$

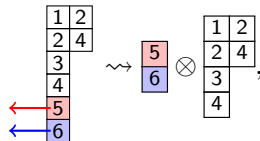


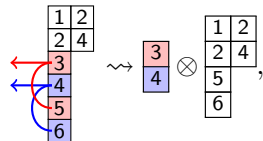
Pieri inclusions removing two boxes

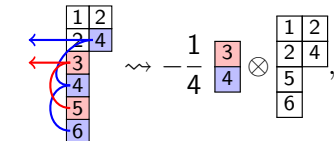


$$\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} \rightsquigarrow \begin{array}{|c|} \hline \color{red}\square \\ \hline \color{blue}\square \\ \hline \end{array} \otimes \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} :$$

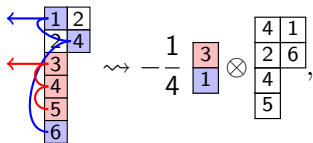
$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & 4 \\ \hline 3 & \\ \hline 4 & \\ \hline \color{red}5 & \\ \hline \color{blue}6 & \\ \hline \end{array} \mapsto \sum_{2\text{-paths } P} \frac{(-1)^{|P|}}{H(P)} \begin{array}{|c|} \hline \color{red}P_2 \\ \hline \color{blue}P_1 \\ \hline \end{array} \otimes P \left(\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} \right)$$



$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & 4 \\ \hline 3 & \\ \hline 4 & \\ \hline \color{red}5 & \\ \hline \color{blue}6 & \\ \hline \end{array} \rightsquigarrow \begin{array}{|c|} \hline \color{red}5 \\ \hline \color{blue}6 \\ \hline \end{array} \otimes \begin{array}{|c|} \hline 1 & 2 \\ \hline 2 & 4 \\ \hline 3 & \\ \hline 4 & \\ \hline \end{array},$$


$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & 4 \\ \hline \color{red}3 & \\ \hline \color{blue}4 & \\ \hline 5 & \\ \hline 6 & \\ \hline \end{array} \rightsquigarrow \begin{array}{|c|} \hline \color{red}3 \\ \hline \color{blue}4 \\ \hline \end{array} \otimes \begin{array}{|c|} \hline 1 & 2 \\ \hline 2 & 4 \\ \hline 5 & \\ \hline 6 & \\ \hline \end{array},$$


$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & 4 \\ \hline \color{red}3 & \\ \hline \color{blue}4 & \\ \hline 5 & \\ \hline 6 & \\ \hline \end{array} \rightsquigarrow -\frac{1}{4} \begin{array}{|c|} \hline \color{red}3 \\ \hline \color{blue}4 \\ \hline \end{array} \otimes \begin{array}{|c|} \hline 1 & 2 \\ \hline 2 & 4 \\ \hline 5 & \\ \hline 6 & \\ \hline \end{array},$$



$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & 4 \\ \hline \color{red}3 & \\ \hline \color{blue}4 & \\ \hline 5 & \\ \hline 6 & \\ \hline \end{array} \rightsquigarrow -\frac{1}{4} \begin{array}{|c|} \hline \color{red}3 \\ \hline \color{blue}1 \\ \hline \end{array} \otimes \begin{array}{|c|c|} \hline 4 & 1 \\ \hline 2 & 6 \\ \hline 4 & \\ \hline 5 & \\ \hline \end{array},$$

Complexity of the descriptions

In both descriptions, the number of terms in the Pieri inclusion acting on $\lambda = (w_1^{h_1}, \dots, w_N^{h_N})$ depends on the number of paths on the diagram.

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Olver's description: paths given by the number of choices of rows in the diagram, where you must choose the first row.

$$2^{h_1-1} \cdot \prod_{i=2}^N 2^{h_i},$$

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Olver's description: paths given by the number of choices of rows in the diagram, where you must choose the first row.

$$2^{h_1-1} \cdot \prod_{i=2}^N 2^{h_i},$$

New description: paths given by the number of choices of rows in the diagram, where you must choose the first row *and cannot skip rows within blocks*.

$$h_1 \cdot \prod_{i=2}^N (h_i + 1).$$

Computation Time Examples

Computation Time Examples

Olver's alg:

```
i3 : time pieri({6,6,6}, {3}, CC[a,b,c])  
    -- used 0.723831 seconds
```

```
o3 = | 6c   |  
     | -36b |  
     | 216a |
```

New alg:

```
i31 : time pieri({6,6,6}, {3}, CC[a,b,c])  
     -- used 0.0219184 seconds
```

```
o31 = | 6c   |  
      | -36b |  
      | 216a |
```

Computation Time Examples

Olver's alg:

```
i4 : time pieri({7,7,7}, {3}, CC[a,b,c])  
    -- used 88.0308 seconds
```

```
o4 = | 7c   |  
     | -49b |  
     | 343a |
```

New alg:

```
i32 : time pieri({7,7,7}, {3}, CC[a,b,c])  
     -- used 0.0350425 seconds
```

```
o32 = | 7c   |  
      | -49b |  
      | 343a |
```

Computation Time Examples

Olver's alg:

```
i3 : time pieri({8,8,8}, {3}, CC[a,b,c])  
^C ^Cstdio:3:6:(3): error: interrupted  
-- used 3538.41 seconds
```

New alg:

```
i33 : time pieri({8,8,8}, {3}, CC[a,b,c])  
-- used 0.0688948 seconds
```

```
o33 = | 8c   |  
      | -64b |  
      | 512a |
```

Thank You!

