Explicit Pieri Inclusions

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Integer partitions

Definition

A **partition** of $n \in \mathbb{N}$ is a sequence of integers

$$\lambda = (\lambda_1, \lambda_2, \ldots)$$

where $\quad \lambda_1 \geq \lambda_2 \geq \cdots \geq 0 \quad$ and $\quad \sum \lambda_i = n.$

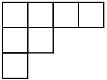
- Only finitely many $\lambda_i \neq 0$
- Each $\lambda_i \neq 0$ is a *part* of λ
- $\lambda \vdash n$ or $|\lambda| = n$

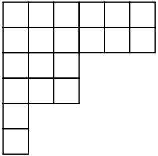
$$\lambda = (4, 2, 1, 0, 0, \ldots) = (4, 2, 1, 0, 0) = (4, 2, 1) \qquad |\lambda| = 7$$

Young diagrams

Partitions of n can be represented by a *Young diagram* of size n, an array of n left-justified boxes with weakly decreasing row length.

Examples:

$\lambda = (4, 2, 1) \quad \longleftrightarrow \quad \lambda =$ 

$\sigma = (6, 6, 3, 3, 1, 1) \quad \longleftrightarrow \quad \sigma =$ 

Blocks of a diagram

Block notation: $\lambda = (w_1^{h_1}, w_2^{h_2} \ldots, w_{N-1}^{h_{N-1}}, w_N^{h_N})$ where $w_i < w_{i+1}$ and each w_i appears as a part of λ exactly h_i times.

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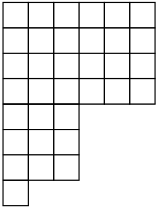
$(6, 6, 6, 6, 3, 3, 3, 1) = (1^1, 3^3, 6^4)$:



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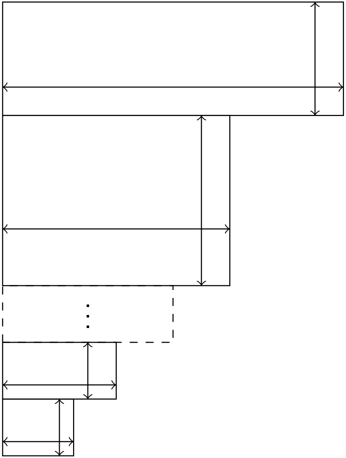

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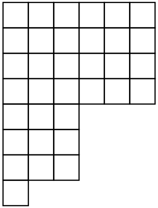

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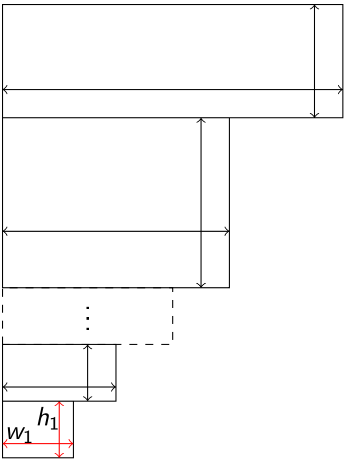

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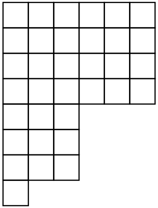

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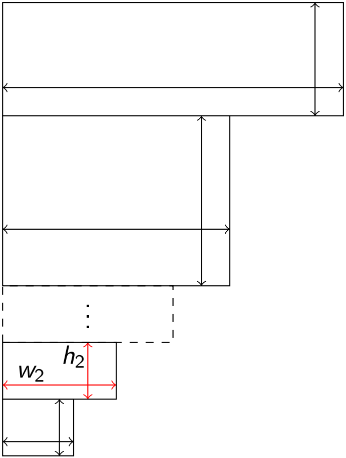

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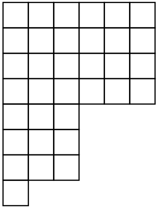

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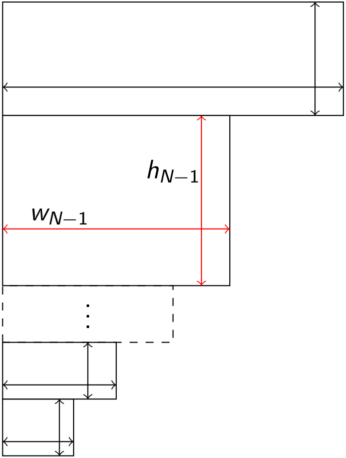

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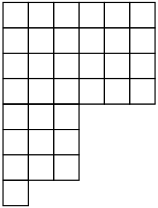

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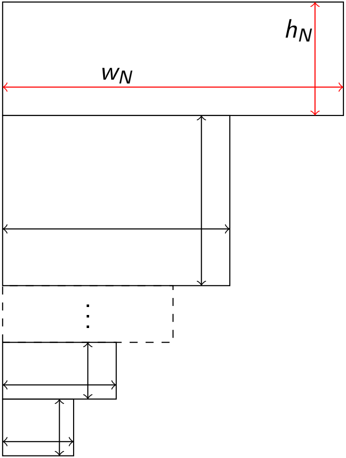

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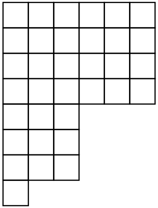

Young tableaux

A *Young tableaux* is a filling of a Young diagram, e.g.

4	2	9	1
7	5		
1			

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Semi-standard ones are weakly increasing across rows and strictly increasing down columns.

1	1	1	1	1	1
2	2	3	4	4	4
3	3	6			
4	5	7			
5					
8					

Representations of $\mathrm{GL}_n(\mathbb{C})$

$$\{\text{polynomial irreducible representations of } \mathrm{GL}_n(\mathbb{C})\}$$

$$\Big\updownarrow \text{ bij}$$

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Weyl modules are the reps of $\mathrm{GL}_n(\mathbb{C})$ via this identification:

$$\lambda \longleftrightarrow \mathbb{S}_\lambda.$$

If $|\lambda| = m$,

- $\mathbb{S}_\lambda$ can be constructed as subspaces (or quotients) of $(\mathbb{C}^n)^{\otimes m}$
- A basis is given by the semi-standard tableaux on λ with entries $1, \ldots, n$

Weyl Modules

E.g.

$$\mathbb{S}_{(m)} = \square\square\cdots\square\square = \mathrm{Sym}^m(\mathbb{C}^n)$$

$$\mathbb{S}_{(1,\ldots,1)} = \begin{array}{c}\square\\\square\\\vdots\\\square\\\square\end{array} = \bigwedge^m(\mathbb{C}^n)$$

The Pieri Rule

Theorem (Pieri Rule)

Let μ be a partition and $\nu = (1, \ldots, 1)$ be a partition of m. Then we have an isomorphism of $\mathsf{GL}_n(\mathbb{C})$*-modules*

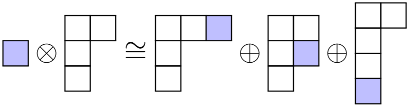
$$\mathbb{S}_\nu \otimes \mathbb{S}_\mu \cong \bigoplus_\lambda \mathbb{S}_\lambda$$

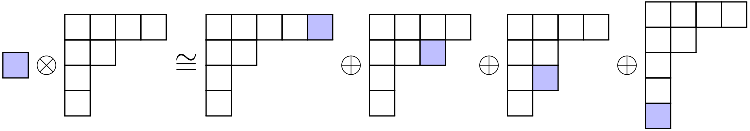
where the sum is over all $\lambda \supset \mu$ obtained by adding m boxes to μ with no two boxes in the same row. Similarly,

$$\mathbb{S}_{(m)} \otimes \mathbb{S}_\mu \cong \bigoplus_\lambda \mathbb{S}_\lambda$$

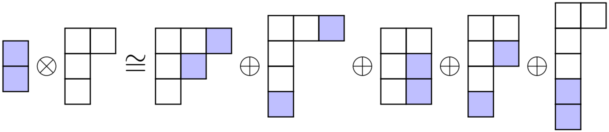
where the sum is over all $\lambda \supset \mu$ obtained by adding m boxes to μ with no two boxes in the same column.

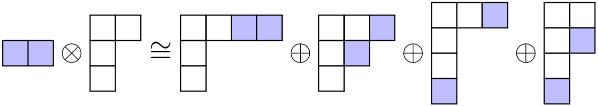
The Pieri Rule - One Box



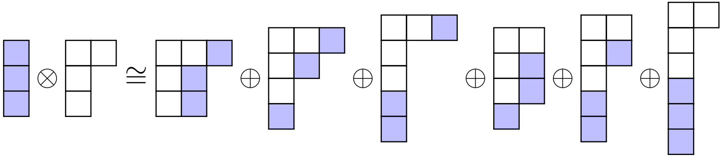


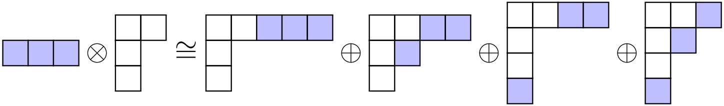
The Pieri Rule - Two Boxes





The Pieri Rule - Three Boxes





Pieri Inclusions

From the Pieri rule we get maps

$$S_\lambda \to S_\nu \otimes S_\mu,$$

called **Pieri inclusions**, unique up to non-zero scalar multiple.

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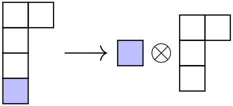

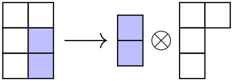

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The first explicit description of such inclusions was given by Olver (1982) in the "1-box removal" case

$$S_\lambda \xrightarrow{\Phi_O} S_{(1)} \otimes S_\mu$$

with the general case given by iteration.

Olver's description of Pieri inclusions

$$\Phi_O = \sum_J \frac{(-1)^{|J|} J}{c_J}$$

The J are the ways to remove the indicated box up and out of the diagram, $|J|$ is the number of rows used, and the c_J depend on the rows used.

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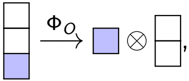

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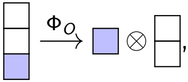
E.g., 

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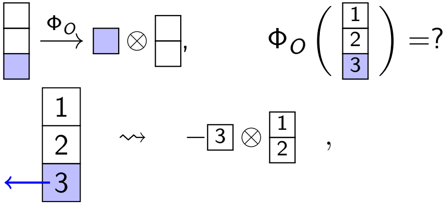

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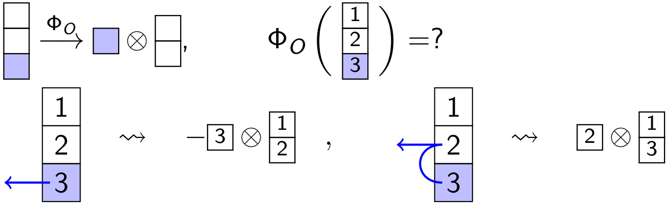

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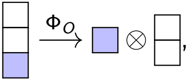
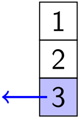
E.g.,  $\quad \Phi_O \left(\begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array} \right) = ?$



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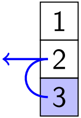
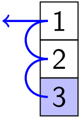


$\rightsquigarrow \quad \boxed{2} \otimes \begin{array}{|c|} \hline 1 \\ \hline 3 \\ \hline \end{array}$



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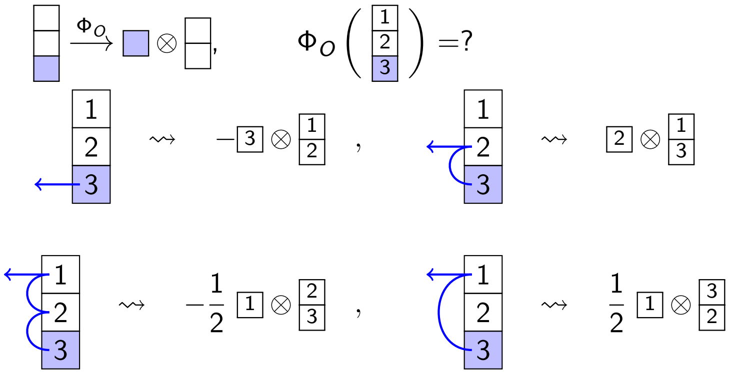

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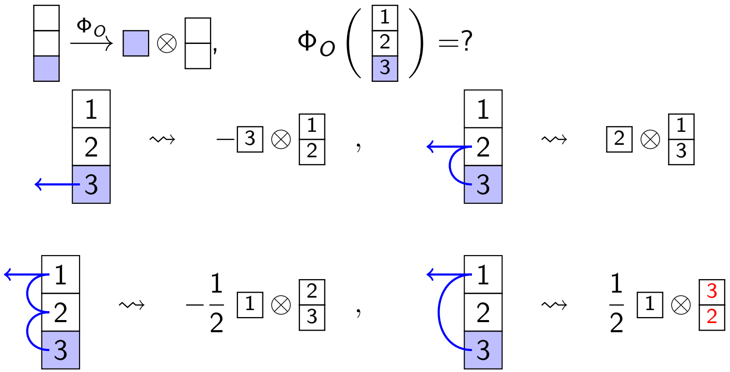

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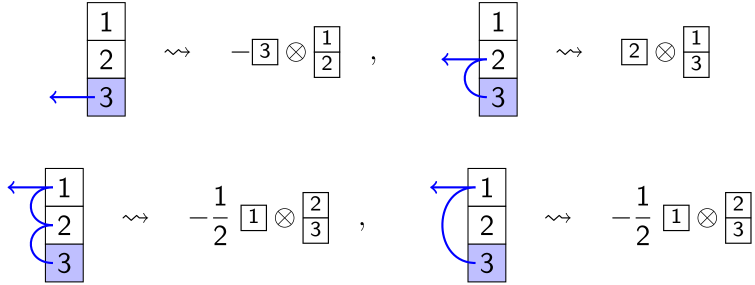

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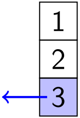

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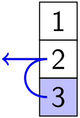

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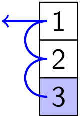

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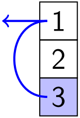

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New description of Pieri inclusions

$$\Phi = \sum_P \frac{(-1)^{|P|} P}{H(P)}$$

The P are the ways to remove the indicated box up and out of the diagram *with a row skipping restriction*. The coefficients $H(P)$ are similar to the c_J, but depend only on the blocks used.

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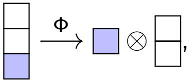
E.g., 

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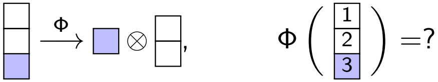

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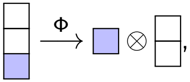

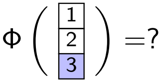

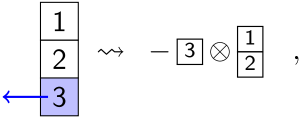

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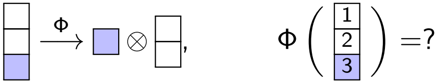

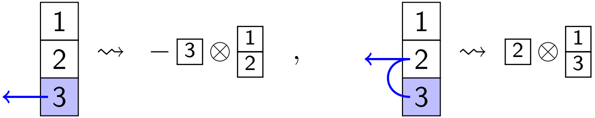

$$\Phi = \sum_{P} \frac{(-1)^{|P|} P}{H(P)}$$

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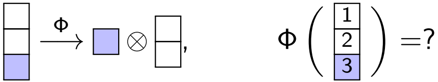

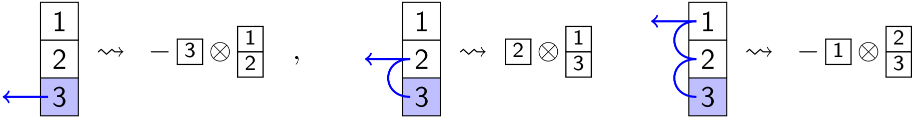

New description of Pieri inclusions

$$\Phi = \sum_{P} \frac{(-1)^{|P|} P}{H(P)}$$

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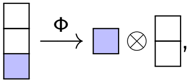
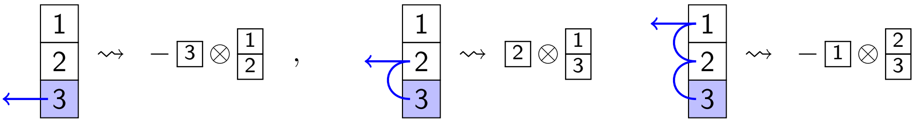
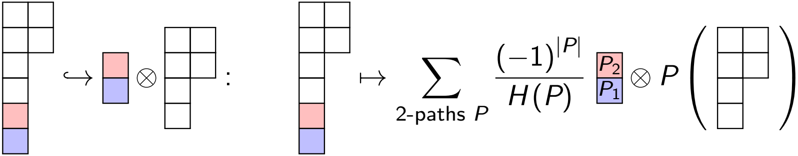
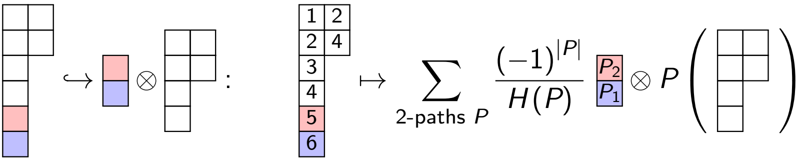


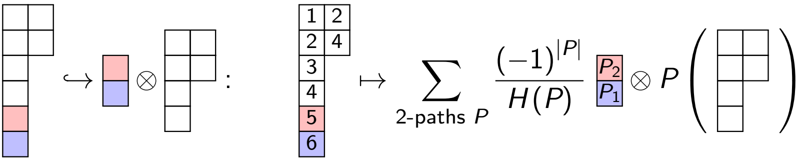
$$\Phi\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\right) = -\boxed{3} \otimes \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} + \boxed{2} \otimes \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} - \boxed{1} \otimes \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}$$



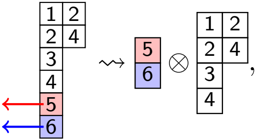

$\hookrightarrow$
$\otimes$
:
$\mapsto$
$\sum_{\text{2-paths } P} \frac{(-1)^{|P|}}{H(P)}$
P_2
P_1
$\otimes P$

Pieri inclusions removing two boxes



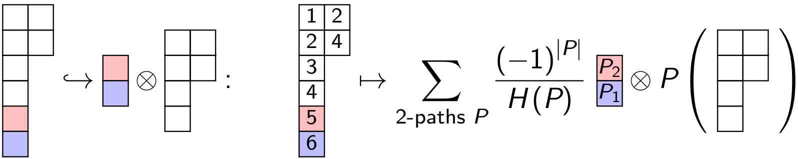


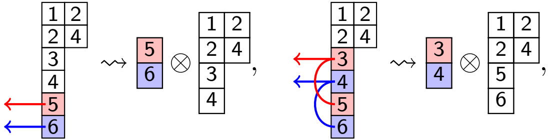
1 2
2 4
3
4
5
6
(−1)^|P|
H(P)
2-paths P
P2
P1
P



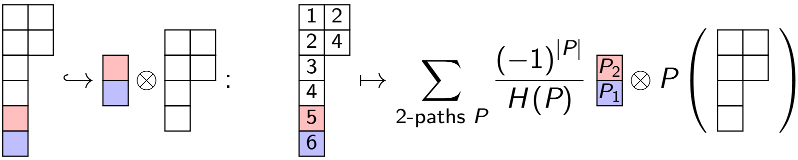
1 2
2 4
3
4
5
6
5
6
1 2
2 4
3
4

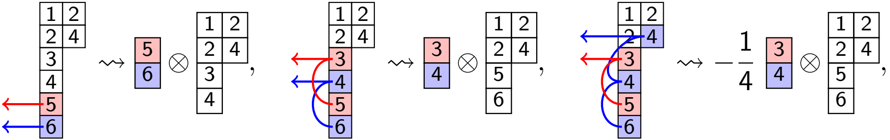
Pieri inclusions removing two boxes



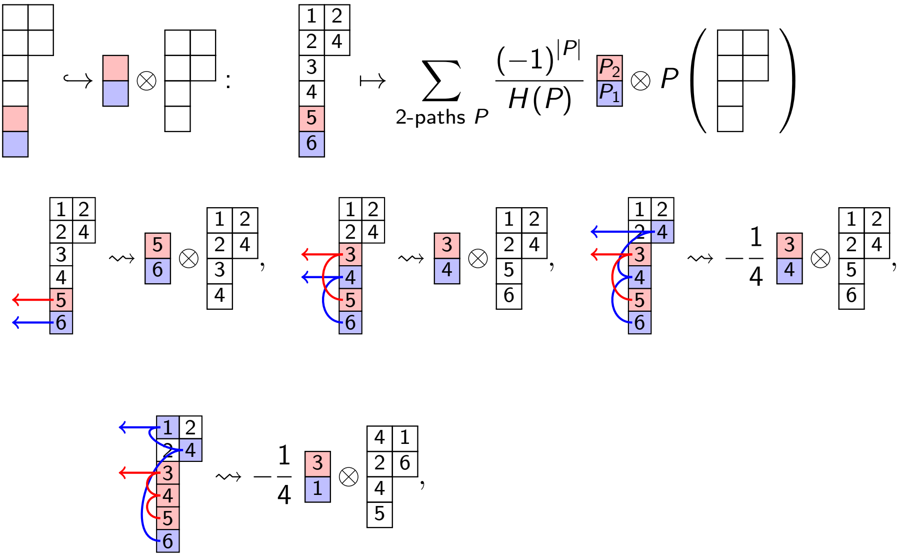


Pieri inclusions removing two boxes

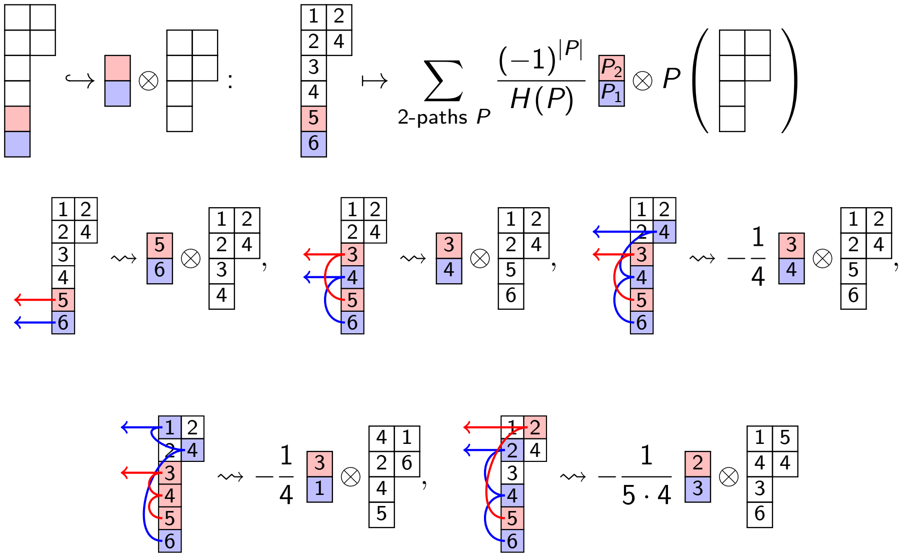




Pieri inclusions removing two boxes



Pieri inclusions removing two boxes



Complexity of the descriptions

In both descriptions, the number of terms in the Pieri inclusion acting on $\lambda = (w_1^{h_1}, \ldots, w_N^{h_N})$ depends on the number of paths on the diagram.

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Olver's description: paths given by the number of choices of rows in the diagram, where you must choose the first row.

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Olver's description: paths given by the number of choices of rows in the diagram, where you must choose the first row.

$$2^{h_1 - 1} \cdot \prod_{i=2}^{N} 2^{h_i},$$

New description: paths given by the number of choices of rows in the diagram, where you must choose the first row *and cannot skip rows within blocks*.

$$h_1 \cdot \prod_{i=2}^{N} (h_i + 1) \, .$$

Computation Time Examples

Computation Time Examples

Olver's alg:

```
i3 : time pieri({6,6,6}, {3}, CC[a,b,c])
     -- used 0.723831 seconds

o3 = | 6c   |
     | -36b |
     | 216a |
```

New alg:

```
i31 : time pieri({6,6,6}, {3}, CC[a,b,c])
     -- used 0.0219184 seconds

o31 = | 6c   |
      | -36b |
      | 216a |
```

Computation Time Examples

Olver's alg:

```
i4 : time pieri({7,7,7}, {3}, CC[a,b,c])
     -- used 88.0308 seconds

o4 = | 7c   |
     | -49b |
     | 343a |
```

New alg:

```
i32 : time pieri({7,7,7}, {3}, CC[a,b,c])
     -- used 0.0350425 seconds

o32 = | 7c   |
      | -49b |
      | 343a |
```

Computation Time Examples

Olver's alg:

```
i3 : time pieri({8,8,8}, {3}, CC[a,b,c])
  ^C ^Cstdio:3:6:(3): error: interrupted
     -- used 3538.41 seconds
```

New alg:

```
i33 : time pieri({8,8,8}, {3}, CC[a,b,c])
     -- used 0.0688948 seconds

o33 = | 8c   |
      | -64b |
      | 512a |
```

Thank You!