Derivation Operators for a Family of Quiver Algebras

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Definition

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- A is a ring with 1 and has a k-vector space structure over.
- The multiplication $A \times A \to A$ on A is compatible with the multiplication in the field. i.e

$$\lambda(ab) = (a\lambda)b = a(\lambda b) = (ab)\lambda$$

for $\lambda \in k$, $a, b \in A$

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- Both $Der_k(A), Der_k(A, M)$ are k-modules. That is $\alpha D, D_1 + D_2 \in Der_k(A, M)$ for all $D, D_1, D_2 \in Der_k(A, M)$.

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  $\alpha D, D_1 + D_2 \in Der_k(A, M)$
  for all $D, D_1, D_2 \in Der_k(A, M)$.
- $Der_k(A, M)$ is a Lie algebra with a Lie bracket

$$[D_1, D_2] = D_1 \cdot D_2 - D_2 \cdot D_1$$

Examples

- Let $C^{\infty}([a, b])$ be the space of all infinitely differentiable functions on the interval $[a, b]$, then

$$D : C^{\infty}([a, b]) \to C^{\infty}([a, b])$$

defined by $D(f) = \frac{d}{dx}(f)$ is a derivation.

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- Let $C(\mathbb{R}^n)$ be the algebra of all real-valued differentiable function on $\mathbb{R}^n$, then

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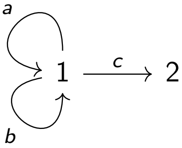
defined by $D(f) = \frac{\partial}{\partial x_i}(f)$ is a derivation on $C(\mathbb{R}^n)$.
- Let A be a non-commutative algebra, Let $a \in A$ be fixed, then
$$D_a(-) : A \to A$$
defined by $D_a(x) = [a,x] = ax - xa$ is a derivation on A.

Examples contd. [T.Oke]

Let Q be the quiver:



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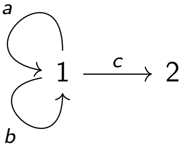

and consider the following family of quiver algebras

$$\Lambda_q = kQ/I \quad where\ I = \langle a^2, b^2, ab - qba, ac \rangle, q \in k \tag{1}$$

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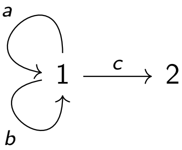

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$$D_{a,a} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} \qquad Der_k(\Lambda_q)/T \cong HH^1(\Lambda_q)$$

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More generally, we can extend D from a projective bi-module resolution $\mathbb{P}$ of A to itself.

Lemma [N.S. Gopalakrishnan and R. Sridharan]
Let $D : A \to A$ be a derivation. Then there are k-linear chain maps

$$\tilde{D}_\bullet : \mathbb{P}_\bullet \to \mathbb{P}_\bullet$$

lifting f with the property

$$\tilde{D}_n((a \otimes b) \cdot x) = D(a)xb + a\tilde{D}_n(x)b + axD(b) \qquad (2)$$

for each n with $a, b \in A$ and $x \in \mathbb{P}_n$. Moreover $\tilde{D}_n$ is unique up to chain homotopy.

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These chain $\tilde{D}_n$ maps are also called **derivation operators**.

Example

Let

$$\mathbb{B}_\bullet := \quad \cdots \to A^{\otimes(n+2)} \xrightarrow{\delta_n} A^{\otimes(n+1)} \to \cdots \to A^{\otimes 3} \xrightarrow{\delta_1} A^{\otimes 2} \to 0$$

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where the differential δ_n are given by

$$\delta_n(a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}$$

and the homology in degree 0 is A.

$\mathbb{B}_\bullet$ is called the bar resolution of A.

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for all $a_0, \cdots, a_{n+1} \in A$, then extend k-linearly.

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for all $a_0, \cdots, a_{n+1} \in A$, then extend k-linearly. Then $\tilde{D}_n$ is a derivation operator satisfying equation (2).
Notice that for $a \otimes b \in A^e$, and $x \in \mathbb{B}_n$

$$\tilde{D}_n((a \otimes b) \cdot x) \neq (a \otimes b)\tilde{D}_n(x)$$

Brackets on Hochschild Cohomology

The Hochschild cohomology of A with coefficients in M is given as

$$HH^*(A) = Ext^*_{A^e}(A, M) = \bigoplus_{n=0} H^n(Hom_k(A^{\otimes n}, M))$$

Lie bracket on $HH^*(A)$

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$$[f, g] = f \circ g - (-1)^{(m-1)(n-1)} g \circ f$$

$f \circ g = \sum_{j=1}^{m} (-1)^{(n-1)(j-1)} f \circ_j g$ where

$$f \circ_j g(a_1 \otimes \cdots a_{m+n-1}) = f(a_1 \otimes \cdots \otimes a_{j-1} \otimes g(a_j \otimes \cdots \otimes a_{j+n-1}) \otimes a_{j+n} \otimes \cdots \otimes a_{m+n})$$

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- Graded Lie bracket: $\delta^*([f, g]) = (-1)^{(n-1)}[\delta^*(f), g] + [f, \delta^*(g)]$

Theorem [M. Suarez-Alvarez]
Let $f : A \to A$ be a derivation and $g \in Hom_k(\mathbb{P}_n, A)$ be any cocycle. Let $\tilde{f}_\bullet : \mathbb{P}_\bullet \to \mathbb{P}_\bullet$ be derivation operators satisfying equation (2). The Gerstenhaber bracket of f and g is given by the following

$$[f, g] = fg - g\tilde{f}_n$$

as cocycles on $\mathbb{P}_n$.

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Compare with $[D_1, D_2] = D_1 \cdot D_2 - D_2 \cdot D_1$.

proof uses chain maps between $\mathbb{B}_\bullet$ and $\mathbb{P}_\bullet$.

Recall previous examples contd.

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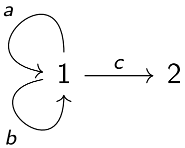

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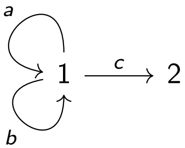

$$Der_k(\Lambda_q) = \langle D_{a,a}, D_{b,b}, D_{c,c}, D_{a,ab}, D_{b,ab}, D_{c,bc} \rangle.$$

Proposition [T.Oke]
Let $\Lambda_q = \frac{kQ}{I}$ be a family of quiver algebra. Let $D : \Lambda_q \to \Lambda_q$ be a derivation on Λ_q. Then the derivation operators $\tilde{D_n} : \mathbb{K}_n \to \mathbb{K}_n$ are defined in the following ways

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- $\tilde{D}_n(\varepsilon^n_r) = t(n,r)\varepsilon^n_r, \qquad$ for some $\quad t(n,r) \in k$.
  if D is any of $\{D_{a,a}, D_{b,b}, D_{c,c}\}$.

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- $\tilde{D}_n(\varepsilon_r^n) = t(n,r)\varepsilon_r^n, \qquad \text{for some} \quad t(n,r) \in k.$
  if D is any of $\{D_{a,a}, D_{b,b}, D_{c,c}\}$.
- $\tilde{D}_n(\varepsilon_r^n) = \begin{cases} tf_k^1\varepsilon_r^n + t'\varepsilon_{r-1}^n f_{k+1}^1, & \text{for } 0 \leq r < n+1 \\ tf_k^1\varepsilon_{n+1}^n + t'\varepsilon_1^n f_{k+1}^1, & \text{whenever } r = n+1. \end{cases}$
  if D is any of $\{D_{a,ab}, D_{b,ab}, D_{c,bc}\}$.

For instance

- If $D = D_{a,a}$, then
$$\tilde{D}_n(\varepsilon_r^n) = \begin{cases} (n-r)\varepsilon_r^n & \textit{when } r = 0,1,2,\cdots,n \\ (n-1)\varepsilon_r^n & \textit{when } r = n+1 \end{cases}.$$

$\phi = (a,0,0)$ ε_r^n $(n\downarrow, r\rightarrow)$	0	1	2	3	4	5	6	$\cdots$	r
0	0	0							
1	$1\varepsilon_0^1$	0	0						
2	$2\varepsilon_0^2$	$1\varepsilon_1^2$	0	$1\varepsilon_3^2$					
3	$3\varepsilon_0^3$	$2\varepsilon_1^3$	$1\varepsilon_2^3$	0	$2\varepsilon_4^3$				
4	$4\varepsilon_0^4$	$3\varepsilon_1^4$	$2\varepsilon_2^4$	$1\varepsilon_3^4$	0	$3\varepsilon_5^4$			
5	$5\varepsilon_0^5$	$4\varepsilon_1^5$	$3\varepsilon_2^5$	$2\varepsilon_3^5$	$1\varepsilon_4^5$	0	$4\varepsilon_6^5$		

Thanks for listening.

Bibliography I

- N. Snashall, *Support varieties and the Hochschild cohomology ring modulo nilpotence*, Proceedings of the 41st Symposium on Ring Theory and Representation Theory, Tsukuba, 2009, pp. 6882
- N. Snashall, . Solberg, *Support varieties and the Hochschild cohomology rings*, Proc. London Maths. Soc. 88 (2004), no. 3, 705732
- R.O. Buchweitz, E.L. Green, N. Snashall, . Solberg, *Multiplicative structures for Koszul algebras*, The Quarterly Journal of Mathematics 2008 59(4), 441-454 , Database: arXiv
- R.O. Buchweitz, E.L. Green, D. Madsenl, . Solberg, *Hochshchild cohomology without finite global dimension*, Math. Res. Lett., vol. 12 (2005), no. 6, 805-816, Database: arXiv

Bibliography II

- E. L. Green, G. Hartman, E. N. Marcos, . Solberg, *Resolutions over Koszul algebras* arXiv:math/0409162
- S. Witherspoon, *Hochschild Cohomology for Algebras*, Graduate Studies in Mathematics, American Mathematical Society, to appear.
- R. Martinez-Villa, *Introduction to Koszul algebras*, Rev. Un. Mat. Argentina 58 (2008), no.2, 67-95.
- F. Xu, *Hochschild and ordinary cohomology rings of small categories*, Adv Math.219 (2008), 1872-1893.
- M. Swarez-Alvarez, *A little bit of Extra functoriality for Ext and the computation of the Gerstenhaber brackets*, arXiv:1604.06507.