Generalities on Hopf algebras

Recall that $\mathcal{U} = \mathcal{U}_q(sl_2) = k[E, F, K, K^{-1}]/R$
where R is the ideal generated by

$$KK^{-1} = 1 = K^{-1}K \qquad (R1)$$

$$KEK^{-1} = q^2 E \qquad (R2)$$

$$KFK^{-1} = q^{-2}F \qquad (R3)$$

$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}} \qquad (R4)$$

$(q \in k \setminus \{0, \pm 1\})$

We've seen a description for the representations of $\mathcal{U}$ both when q is a root of 1 and otherwise.

For a deeper study of $\mathrm{Rep}(\mathcal{U})$ as well as the representation theory of quantum groups other than $\mathcal{U}_q(sl_2)$, we need a feature of the representation category we have yet to discuss: the ability to tensor 2 representations.

This feature is already present in rep theory of groups and Lie algebras: if G is a group and $U, V \in \mathrm{Rep}(G)$, then $U \otimes V \in \mathrm{Rep}(G)$ by

$$g \cdot (u \otimes v) = g \cdot u \otimes g \cdot v$$

and if $U, V \in \mathrm{Rep}(\mathfrak{g})$ for a Lie alg $\mathfrak{g}$ then

$U \otimes V \in \mathrm{Rep}(\mathfrak{g})$ by

$$x \cdot (u \otimes v) = u \otimes x \cdot v + x \cdot u \otimes v.$$

In fact, this happens because

$$\mathrm{Rep}(G) \cong \mathrm{Rep}(kG)$$
$$\mathrm{Rep}(\mathfrak{g}) \cong \mathrm{Rep}(\mathcal{U}\mathfrak{g})$$

and both kG and $\mathcal{U}\mathfrak{g}$ are Hopf algebras.

Quick intro to Hopf algebras

A Hopf algebra is a bialgebra which admits an antipode.

Quick intro to bialgebras

Given an k-algebra A, we can consider $\mathrm{Rep}(A)$.
A bialgebra is an algebra with additional structures s.t. these structures correspond to $\mathrm{Rep}(A)$ being a monoidal category

(roughly speaking)

a category $\mathcal{C}$ with a product $X \otimes Y \in \mathcal{C}$ $\forall X, Y \in \mathcal{C}$ and unit $\mathbb{1} \in \mathcal{C}$ s.t.

$$(X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z)$$
$$X \otimes 1 \cong X \cong 1 \otimes X$$

For $\mathrm{Rep}(A)$ to be a monoidal cat, we need for all $U, V \in \mathrm{Rep}(A)$, $U \otimes V \in \mathrm{Rep}(A)$ s.t.

$$(U \otimes V) \otimes W \cong U \otimes (V \otimes W) \text{ as } A\text{-modules}$$

and a $\mathrm{Rep}(A)$ structure on k s.t.

$$U \otimes k \cong U \cong k \otimes U \text{ as } A\text{-modules.}$$

We know that if A, B are k-algebras, then a k-alg map $f : A \to B$ induces a functor $f^* : \mathrm{Rep}(B) \longrightarrow \mathrm{Rep}(A)$.

If $U, V \in \mathrm{Rep}(A)$, then $U \otimes V \in \mathrm{Rep}(A \otimes A)$ so a natural way to have $U \otimes V \in \mathrm{Rep}(A)$ is to have a k-alg map $A \xrightarrow{\Delta} A \otimes A$.

Similarly, a natural way to have $k \in \mathrm{Rep}(A)$ is to have a k-alg map $A \xrightarrow{\varepsilon} k$.

The condition that $\forall\, U, V, W \in \mathrm{Rep}(A)$,

$$(U \otimes V) \otimes W \cong U \otimes (V \otimes W) \text{ as } A\text{-modules}$$

translates to comm. diagram

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & A \otimes A \arrow[d, "\Delta \otimes \mathrm{id}"] \\ A \otimes A \arrow[r, "\mathrm{id} \otimes \Delta"'] & A \otimes A \otimes A \end{tikzcd} \qquad (1)$$

and the condition that $A \otimes k \cong A \cong k \otimes A$ as

A-modules translates to comm. diagram

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "\Delta"'] \arrow[dr, "id"] & A \otimes A \arrow[d, "id \otimes \varepsilon"] \\ A \otimes A \arrow[r, "\varepsilon \otimes id"'] & A \end{tikzcd} \qquad (2)$$

<u>Def</u>. A bialgebra is a k-algebra A with two algebra maps (Δ, ε) satisfying (1) and (2).
(A, Δ, ε) where $A \in Vect$ is called a coalgebra.

Δ : coproduct (1) A is coassociative

ε : counit (2) A is counital

<u>Fact</u>. If A is an algebra $A = (A, m, u)$
and A is a coalgebra $A = (A, \Delta, \varepsilon)$
then A is a bialgebra
$\Leftrightarrow$ Δ and ε are algebra maps
$\Leftrightarrow$ m and u are coalgebra maps

<u>What about the antipode ?</u>

If $V \in Rep(G)$, then $V^* = Hom(V, k) \in Rep(G)$ by

$$(g \cdot \varphi)(v) = \varphi(g^{-1} \cdot v) \qquad g \in G, \varphi \in V^*, v \in V.$$

Similarly, if $V \in Rep\, \mathfrak{g}$ then $V^* \in Rep\, \mathfrak{g}$ by

$$(x \cdot \varphi)(v) = \varphi((-x) \cdot v) \qquad x \in \mathfrak{g}, \varphi \in V^*, v \in V.$$

So Rep G and Rep $\mathfrak{g}$ have the extra properties that objects have "duals".
In a general monoidal category, $\mathrm{Hom}_{\mathcal{C}}(X, \mathbb{1})$ is not an object of $\mathcal{C}$. Instead, we make the following definition:

Def. Let $\mathcal{C} = (\mathcal{C}, \otimes, \mathbb{1})$ be a monoidal category.
A duality in $\mathcal{C}$ is a 4-tuple

$$(X, Y, \mathrm{ev}: X \otimes Y \to \mathbb{1}, \mathrm{coev}: \mathbb{1} \to Y \otimes X)$$

s.t. the following maps are identity maps:

$$X \xrightarrow{\cong} X \otimes \mathbb{1} \xrightarrow{\mathrm{id} \otimes \mathrm{coev}} X \otimes Y \otimes X \xrightarrow{\mathrm{ev} \otimes \mathrm{id}} \mathbb{1} \otimes X \xrightarrow{\cong} X$$

$$Y \xrightarrow{\cong} \mathbb{1} \otimes Y \xrightarrow{\mathrm{coev} \otimes \mathrm{id}} Y \otimes X \otimes Y \xrightarrow{\mathrm{id} \otimes \mathrm{ev}} Y \otimes \mathbb{1} \xrightarrow{\cong} Y$$

In this case, we say that Y is a right dual of X and X is a left dual of Y.

Eg. $\mathcal{C} = \mathrm{Vect}_k$. For any $V \in \mathrm{Vect}_k^{fd}$, the maps

$$\mathrm{ev}: V^* \otimes V \to k, \quad (\varphi, v) \mapsto \varphi(v)$$

$$\mathrm{coev}: k \to V \otimes V^*, \quad 1 \mapsto \sum v_i \otimes v^i$$

make $(V^*, V, \mathrm{ev}, \mathrm{coev})$ a duality.
if $\widetilde{\mathrm{ev}}$ and $\widetilde{\mathrm{coev}}$ denote the flipped maps of ev and coev, then $(V, V^*, \widetilde{\mathrm{ev}}, \widetilde{\mathrm{coev}})$ is also a duality.

Def. A rigid monoidal category is a monoidal cat. s.t. every object has a left and a right dual.

If H is a bialgebra, then for any $V \in \mathrm{Rep}(H)^{fd}$, we want $(V^*, V, ev, coev)$ and $(V, V^*, \widetilde{ev}, \widetilde{coev})$ to be dualities in $\mathrm{Rep}(H)^{fd}$, i.e. $\mathrm{Rep}(H)^{fd}$ is a rigid monoidal category. In particular, we want the four morphisms ev, $coev$, $\widetilde{ev}$ and $\widetilde{coev}$ to be H-mod morphisms. This translates to the following conditions for the map S:

(i) S is bijective

(ii) $S(h_1) h_2 = \varepsilon(h) 1 = h_1 S(h_2)$

(iii) $S^{-1}(h_1) h_2 = \varepsilon(h) 1 = h_1 S^{-1}(h_2)$

(automatic from (i) + (ii))

In fact, from (i) + (ii), can show that S is an anti-alg map and anti-coalg map.

$(S : H \to H^{op} \; / \; S : H \to H^{cop})$.

<u>Some remarks.</u>

(1) The antipode map S is unique if it exists, since it is the convolution inverse for the identity map in $\mathrm{Hom}(H, H)$.

(2) Since S is an alg map $H \to H^{op}$, we have an induced functor

$$\mathrm{Mod}\text{-}H = H^{op}\text{-}\mathrm{Mod} \longrightarrow H\text{-}\mathrm{Mod},$$

hence S lets us switch sides.

(3) If φ is an (anti) algebra automorphism on a Hopf algebra $(H, \Delta, \varepsilon, S)$, we can define a new Hopf alg structure $(H, {}^{\varphi}\Delta, {}^{\varphi}\varepsilon, {}^{\varphi}S)$, where

$$ {}^{\varphi}\Delta = (\varphi \otimes \varphi) \circ \Delta \circ \varphi^{-1} $$

$$ {}^{\varphi}\varepsilon = \varepsilon \circ \varphi^{-1} $$

$$ {}^{\varphi}S = \begin{cases} \varphi \circ S \circ \varphi^{-1} & \text{if } \varphi : H \to H \\ \varphi \circ S^{-1} \circ \varphi^{-1} & \text{if } \varphi : H \to H^{op} \end{cases} $$

(4) If H is a Hopf algebra and $M, N \in Rep(H)$, then $Hom(M,N)$ is an H-H bimod by

$$ h \cdot f \cdot k\,(m) = h \cdot f\,(k \cdot m) $$

Via $S : H^{op} \to H$, $Hom(M,N)$ is an H-H^{op} bimod by $h \cdot f \cdot k\,(m) = h \cdot f\,(S(k) \cdot m)$

and since $H\text{-Bimod-}H^{op} = (H \otimes H)\text{-Mod}$, $Hom(M,N) \in Rep(H)$ via the map $\Delta : H \to H \otimes H$:

$$ h \cdot f(m) = h_1 \cdot f\,(S(h_2) \cdot m). \qquad (*) $$

When $N = k$, $M^* = Hom(M,k) \in Rep(H)$.

Further, the linear map

$$ \begin{aligned} N \otimes M^* &\longrightarrow Hom_k(M,N) \\ n \otimes f &\longmapsto \varphi_{f,n}(m) = f(m)\,n \end{aligned} $$

is an H-mod map when M^*, $Hom_k(M,N) \in Rep\,H$ via $(*)$.

For any $P \in Rep(H)$, define

$$ P^H = \{ p \in P \mid h \cdot p = \varepsilon(h)\,p \ \ \forall h \in H \} $$

One can show that

$$ Hom_k(M,N)^H = Hom_H(M,N). $$

$U_q(sl_2)$ as a Hopf algebra

Recall that $\mathcal{U} = U_q(sl_2) = k[E, F, K, K^{-1}]/R$
where R is the ideal generated by

$$KK^{-1} = 1 = K^{-1}K \qquad (R1)$$
$$KEK^{-1} = q^2 E \qquad (R2)$$
$$KFK^{-1} = q^{-2} F \qquad (R3)$$
$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}} \qquad (R4)$$

$(q \in k \setminus \{0, \pm 1\})$

<u>Lemma 3.1.</u> $\exists !$ k-alg map $\Delta : \mathcal{U} \to \mathcal{U} \otimes \mathcal{U}$
defined by
$$\Delta(E) = E \otimes 1 + K \otimes E$$
$$\Delta(F) = F \otimes K^{-1} + 1 \otimes F$$
$$\Delta(K) = K \otimes K.$$

<u>Pf.</u> Let's check $\Delta(R4)$: we need

$$\Delta(E)\Delta(F) - \Delta(F)\Delta(E) \stackrel{?}{=} \frac{\Delta(K) - \Delta(K^{-1})}{q - q^{-1}}$$

$$\begin{aligned}\text{LHS} = {} & EF \otimes K^{-1} + \cancel{E \otimes F} + KF \otimes EK^{-1} + K \otimes EF \\ & - FE \otimes K^{-1} - FK \otimes K^{-1}E - \cancel{E \otimes F} - K \otimes FE\end{aligned}$$

also,
$$\begin{aligned} KF \otimes EK^{-1} &= KF \otimes q^2 K^{-1} E \quad (\text{by } R2) \\ &= q^2 KF \otimes K^{-1}E \\ &= FK \otimes K^{-1}E \quad (\text{by } R3)\end{aligned}$$

So we are left with

$$(EF-FE)\otimes K^{-1} + K\otimes (EF-FE)$$
$$= \big((K-K^{-1})\otimes K^{-1} + K\otimes (K-K^{-1})\big) \,/\, (q-q^{-1}) \quad \text{(by R4)}$$
$$= \text{RHS}$$

For $\Delta(R2)$ and $\Delta(R3)$, we use that $U\otimes U$ has a natural grading induced from U, and prove for each homogeneous component using the formula

$$(K\otimes K^{-1})\, u\, (K^{-1}\otimes K) = q^{2n} u \qquad \forall u \in (U\otimes U)_n.$$

<u>Lemma 3.2</u>. $\Delta : U \to U\otimes U$ is coassociative.

<u>Pf</u>. Straightforward calculation. $\square$

<u>Lemma 3.4</u>. $\exists !$ k-alg map $\varepsilon : U \to k$ s.t.

$$\varepsilon(E) = \varepsilon(F) = 0$$
$$\varepsilon(K) = 1$$

s.t.
$$\begin{tikzcd} U \arrow[r, "\Delta"] \arrow[d, "\Delta"'] \arrow[dr, "id"] & U\otimes U \arrow[d, "id\otimes\varepsilon"] \\ U\otimes U \arrow[r, "\varepsilon\otimes id"'] & U \end{tikzcd}$$
is commutative.

<u>Pf</u>. Same as 3.2. $\square$

<u>Lemma 3.6.</u> $\exists!$ alg map $S : \mathcal{U} \to \mathcal{U}^{op}$ with

$$S(E) = -K^{-1}E, \quad S(F) = -FK \qquad (1)$$
$$S(K) = K^{-1}$$

One has

$$S^2(u) = K^{-1}uK \quad \forall u \in \mathcal{U} \qquad (2)$$

<u>Pf</u>. For (1), we need to check $S(R2) - S(R4)$.

$S(R2)$: $\quad S(K)S(E)S(K^{-1}) \overset{?}{=} q^2 S(E)$

$$\text{LHS} = K^{-1} \cdot^{op} (-K^{-1}E) \cdot^{op} K$$
$$= K(-K^{-1}E)K^{-1} = q^2(-K^{-1}E) = \text{RHS}.$$
$$-K^{-1}E \in \mathcal{U}_1$$

and $KuK^{-1} = q^{2n}u$ for $u \in \mathcal{U}_n$

$S(R3)$ and $S(R4)$ can be checked similarly.

(2) is a very simple check. $\blacksquare$

<u>Rmk 3.10</u> (Quantum trace)

Let $\mathcal{C}$ be a rigid monoidal category, and let's say that we have a natural isomorphism $\alpha_X : X \xrightarrow{\cong} X^{**}$ $\forall X \in \mathcal{C}$. Then we can define $tr_q(X) \in \mathrm{End}(\mathbb{1})$, called the **quantum trace of X** (wrt α), to be

$$\mathbb{1} \xrightarrow{coev_X} X \otimes X^* \xrightarrow{\alpha_X \otimes id} X^{**} \otimes X^* \xrightarrow{ev_{X^*}} \mathbb{1}.$$

In our case, $\mathcal{C} = \mathrm{Rep}(\mathcal{U})^{fd}$. We want an H-mod map $\alpha_M : M \xrightarrow{\cong} M^{**}$. Unfortunately, the linear map $M \xrightarrow{\varphi} M^{**}$, $m \longmapsto \varphi_m(f) = f(m)$ is not H-linear, since $u \cdot \varphi(m) = \varphi(S^2(u) \cdot m) \neq \varphi(u \cdot m)$ in general.

However since $S^2(u) = K^{-1} u K$, we just need to make a slight modification: the map

$$M \xrightarrow{\varphi'} M^{**}, \quad m \longmapsto \varphi'_m(f) = f(K^{-1} m)$$

is now a $\mathcal{U}$-module map.

For a fd-module M, we can identify $\mathrm{End}_k(M) \xrightarrow{\cong} M \otimes M^* \longrightarrow k$ with trace then we can identify the quantum trace tr_q with the map

$$\mathrm{End}_k(M) \longrightarrow k, \quad \varphi \longmapsto \mathrm{tr}(\varphi \circ K^{-1}).$$

Next time: $\mathcal{U}$ as a quasi-triangular Hopf algebra.