# The commutativity constraint for $\mathcal{U}$-modules.

Fix a field $k$ and $q \in k \setminus \{0, \pm 1\}$.

Recall $\mathcal{U} = \mathcal{U}_q(\mathfrak{sl}_2)$:

As a $k$-algebra, $\mathcal{U} = k[E, F, K, K^{-1}] / R$,

$R$ generated by

$$KK^{-1} = 1 = K^{-1}K$$

$$KEK^{-1} = q^2 E$$

$$KFK^{-1} = q^{-2} F$$

$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}$$

Last time, we also discussed a Hopf algebra structure on $\mathcal{U}$: we have algebra maps

$$\Delta : \mathcal{U} \to \mathcal{U} \otimes \mathcal{U} \qquad \qquad \varepsilon : \mathcal{U} \to k$$

$$\Delta(E) = E \otimes 1 + K \otimes E \qquad \qquad \varepsilon(E) = 0$$

$$\Delta(F) = F \otimes K^{-1} + 1 \otimes F \qquad \qquad \varepsilon(F) = 0$$

$$\Delta(K) = K \otimes K \qquad \qquad \varepsilon(K) = 1$$

and an anti-algebra / anti-coalg automorphism

$$S : \mathcal{U} \to \mathcal{U}$$

$$S(E) = -K^{-1}E$$

$$S(F) = -FK$$

$$S(K) = K^{-1}$$

$(\mathcal{U}, \Delta, \varepsilon, S)$ is a Hopf algebra, which makes $\mathrm{Rep}(\mathcal{U})^{fd}$ a rigid monoidal (linear) category.

Also recall that if $\varphi : U \to U$ is an anti-alg automorphism then we can define a new Hopf alg structure $(U, {}^{\varphi}\Delta, {}^{\varphi}\varepsilon, {}^{\varphi}S)$ by

$${}^{\varphi}\Delta = (\varphi \otimes \varphi) \circ \Delta \circ \varphi^{-1} \qquad {}^{\varphi}\varepsilon = \varepsilon \circ \varphi^{-1}$$

$${}^{\varphi}S = \varphi \circ S^{-1} \circ \varphi^{-1}$$

In particular, letting $\varphi = \tau$ the anti-automorphism defined from Chapter 1: $\tau(E) = E$, $\tau(F) = F$, $\tau(K) = K^{-1}$, we get ${}^{\tau}\varepsilon = \varepsilon$ and

$${}^{\tau}\Delta(E) = E \otimes 1 + K^{-1} \otimes E \qquad {}^{\tau}S(E) = -KE$$

$${}^{\tau}\Delta(F) = F \otimes K + 1 \otimes F \qquad {}^{\tau}S(F) = -FK^{-1}$$

$${}^{\tau}\Delta(K) = K \otimes K \qquad {}^{\tau}S(K) = K^{-1}$$

Today, we want to construct a braiding on $\mathrm{Rep}(U)^{fd}$, i.e. natural isomorphisms of f.d. $U$-modules $M \otimes N \xrightarrow{\cong} N \otimes M$ satisfying two hexagon identities. It turns out that we can only accomplish this goal for a subset of $\mathrm{Rep}(U)^{fd}$, using a "generalized R-matrix".

Assume $q$ is not a root of unity and $\mathrm{char}(k) \neq 2$.

<u>Def</u>. Set for all integers $n \geq 0$

$$\Theta_n = a_n F^n \otimes E^n \in U \otimes U$$

where

$$a_n = (-1)^n q^{-n(n-1)/2} \frac{(q-q^{-1})^n}{[n]!} \in k.$$

In particular, $\Theta_0 = 1 \otimes 1$, $\Theta_1 = -(q-q^{-1}) F \otimes E$

$\Theta_{-1} = 0$

$a_n$ satisfies the recursion

$$a_n = -q^{-(n-1)} \frac{q-q^{-1}}{[n]} a_{n-1}.$$

<u>Lemma</u>. $\forall n \geq 0$,

(1) $(E \otimes 1)\Theta_n + (K \otimes E)\Theta_{n-1}$

$= \Theta_n (E \otimes 1) + \Theta_{n-1}(K^{-1} \otimes E)$

(2) $(1 \otimes F)\Theta_n + (F \otimes K^{-1})\Theta_{n-1}$

$= \Theta_n (1 \otimes F) + \Theta_{n-1}(F \otimes K)$

(3) $(K \otimes K)\Theta_n = \Theta_n (K \otimes K)$.

<u>Pf</u>. Part (3) follows from an earlier formula

$$(K \otimes K) u = q^{2n} u (K \otimes K), \qquad u \in (U \otimes U)_n$$

here $\Theta_n = a_n F^n \otimes E^n \in (U \otimes U)_0$. Parts (1) + (2) follow from elementary calculations. $\square$

Let $M$ and $N \in \operatorname{Rep}(U)^{fd}$. Recall that $E$ and $F$ act nilpotently on $M$ and $N$, hence we can define a linear transformation

$$\Theta = \Theta_{M,N} : M \otimes N \longrightarrow N \otimes M$$

by $\Theta = \sum_{n \geq 0} \Theta_n$. (Note $\Theta \notin U \otimes U$).

The formulas from previous lemma imply

$$\Delta(u) \circ \Theta = \Theta \circ {}^{\tau}\Delta(u) \qquad \forall u \in U.$$

Since $F \otimes E$ acts nilpotently on $M \otimes N$, we can find a basis s.t. the matrix of $F \otimes E$ is strictly lower triangular. Each $\Theta_n$ is (up to scalar) equal to $(F \otimes E)^n$, so for $n > 0$ its matrix is strictly upper triangular.

Since $\Theta_0 = \mathrm{id}$ and $\Theta = \sum_{n \geq 0} \Theta_n$ we see that

$\Theta_{M,N}$ is bijective.

Recall that for $M \in \mathrm{Rep}(U)^{fd}$, we have

$$M = \bigoplus_{\lambda \in k} M_\lambda$$

where $M_\lambda = \{ m \in M : Km = \lambda m \}$.

Further, the (non-zero) weights are contained in

$$\tilde{\Lambda} = \{ \pm q^a \mid a \in \mathbb{Z} \}.$$

Suppose we have a map $f : \tilde{\Lambda} \times \tilde{\Lambda} \to k^\times$ s.t.

$$f(\lambda, \mu) = \lambda f(\lambda, \mu q^2) = \mu f(\lambda q^2, \mu) \quad \forall \lambda, \mu \in \tilde{\Lambda}$$

(will see why we want this map soon).

Then we can define, $\forall M, N \in \mathrm{Rep}(U)^{fd}$, a bijective linear transformation $\tilde{f} : M \otimes N \to M \otimes N$ by

$$\tilde{f}(m \otimes m') = f(\lambda, \mu)\, m \otimes n \quad , m \in M_\lambda, \; n \in N_\mu.$$

Set

$$\Theta^f = \Theta \circ \tilde{f}$$

<u>Lemma</u>. $\Delta(u) \circ \Theta^f = \Theta^f \circ (P_{u,u} \circ \Delta)(u)$

Here $P_{V,W} : V \otimes W \to W \otimes V$ denote the switch map $v \otimes w \mapsto w \otimes v$.

Pf. Recall that
$$\Delta(u) \circ \Theta = \Theta \circ {}^{\tau}\Delta(u)$$
$$\Rightarrow \Delta(u) \circ \Theta f = \Theta \circ {}^{\tau}\Delta(u) \circ \tilde{f}$$
and so it suffices TS that
$${}^{\tau}\Delta(u) \circ \tilde{f} = \tilde{f} \circ (P \circ \Delta)(u)$$
Only need to check for generators $E, F, K$, ie.

$$(E \otimes 1 + K^{-1} \otimes E) \circ \tilde{f} = \tilde{f} \circ (E \otimes K + 1 \otimes E) \qquad (1)$$
$$(1 \otimes F + F \otimes K) \circ \tilde{f} = \tilde{f} \circ (K^{-1} \otimes F + F \otimes 1) \qquad (2)$$
$$(K \otimes K) \circ \tilde{f} = \tilde{f} \circ (K \otimes K) \qquad (3)$$

Formula (3) is clear since $\tilde{f}$ stabilizes the weight spaces. First 2 formulas are similar, we'll show (1):
$\forall\, m \in M_\lambda$ and $n \in N_\mu$, $\lambda, \mu \in k$,
$$\text{LHS}(m \otimes n) = f(\lambda, \mu)\left(Em \otimes n + \lambda^{-1} m \otimes En\right)$$
$$\text{RHS}(m \otimes n) = \tilde{f}(Em \otimes \mu n + m \otimes En)$$
$$= f(\lambda q^2, \mu)\, \mu\, Em \otimes n + f(\lambda, \mu q^2)\, m \otimes En$$
(Recall $EM_\lambda \subset M_{q^2\lambda}$, $FM_\lambda \subset M_{q^{-2}\lambda}$.)

Equality follows from
$$f(\lambda, \mu) = \mu f(\lambda q^2, \mu) = \lambda f(\lambda, \mu q^2)$$

Theorem 3.14. Let $M, N \in \mathrm{Rep}(U)^{fd}$. The map
$$\Theta f \circ P : M \otimes N \longrightarrow N \otimes M$$
is a natural isomorphism of $U$-modules.

Pf. Naturality is clear from our construction. The map $\theta^f \circ P$ is linear and bijective because $\theta^f$ and $P$ are so. We have that $\forall u \in U$, $m \in M$, $n \in N$,

$$P(u\cdot(m\otimes n)) = P\circ\Delta(u)(m\otimes n)$$
$$= (P\circ\Delta)(u)\, P(m\otimes n)$$

so $$\theta^f\circ P(u\cdot(m\otimes n)) = (\theta^f\circ P\circ\Delta)(u)\, P(m\otimes n)$$

(by prev. lemma) $$= \Delta(u)\circ\theta^f \quad P(m\otimes n)$$
$$= u\cdot(\theta^f\circ P(m\otimes n))$$

$\boxed{}$

Rmk. The condition of $f : \tilde{\Lambda}\times\tilde{\Lambda}\to k$, $\tilde{\Lambda} = \{\pm q^a \mid a\in\mathbb{Z}\}$ that

$$f(\lambda,\mu) = \mu f(\lambda q^2,\mu) = \lambda f(\lambda,\mu q^2)$$

means that $\forall\, m,n\in\mathbb{Z}$ and $\varepsilon_1,\varepsilon_2\in\{\pm1\}$

$$f(\varepsilon_1 q^{2m},\varepsilon_2 q^{2n}) = \varepsilon_1^n\varepsilon_2^m q^{-2mn} f(\varepsilon_1,\varepsilon_2)$$
$$f(\varepsilon_1 q^{2m+1},\varepsilon_2 q^{2n}) = \varepsilon_1^n\varepsilon_2^m q^{-(2m+1)n} f(\varepsilon_1 q,\varepsilon_2)$$
$$f(\varepsilon_1 q^{2m},\varepsilon_2 q^{2n+1}) = \varepsilon_1^n\varepsilon_2^m q^{-(2n+1)m} f(\varepsilon_1,\varepsilon_2 q)$$
$$f(\varepsilon_1 q^{2m+1},\varepsilon_2 q^{2n+1}) = \varepsilon_1^n\varepsilon_2^m q^{-(2nm+m+n)} f(\varepsilon_1 q,\varepsilon_2 q)$$

Hence $f$ is completely decided by the 16 arbitrary choices of $f$ on the RHS.

For example, if $f(q,q) = q^{-1}$, then the formulas imply that $f(q^{-1},q^{-1}) = q^{-1}$ and $f(q^{-1},q) = 1 = f(q,q^{-1})$.

We want to prove the following theorem.

<u>Main theorem.</u> Let $M, N, P \in \mathrm{Rep}(U)^{fd}$. Suppose that

$$\begin{cases} f(\lambda, \mu\nu) = f(\lambda,\mu)\, f(\lambda,\nu) \\ f(\lambda\mu, \nu) = f(\lambda,\nu)\, f(\mu,\nu) \end{cases}$$

for all weights $\lambda, \mu, \nu$ of these modules. Then the following diagrams commute:

$$\begin{tikzcd}
 & M \otimes (P \otimes N) \arrow[r, phantom, "\cong"] & (M \otimes P) \otimes N \arrow[dr, "R \otimes id"] & \\
M \otimes (N \otimes P) \arrow[ur, "id \otimes R"] \arrow[d, phantom, "\cong"] & & & (P \otimes M) \otimes N \arrow[d, phantom, "\cong"] \\
(M \otimes N) \otimes P \arrow[rrr, "R"] & & & P \otimes (M \otimes N)
\end{tikzcd}$$

$$\begin{tikzcd}
 & (N \otimes M) \otimes P \arrow[r, phantom, "\cong"] & N \otimes (M \otimes P) \arrow[dr, "id \otimes R"] & \\
(M \otimes N) \otimes P \arrow[ur, "R \otimes id"] \arrow[d, phantom, "\cong"] & & & N \otimes (P \otimes M) \arrow[d, phantom, "\cong"] \\
M \otimes (N \otimes P) \arrow[rrr, "R"] & & & (N \otimes P) \otimes M
\end{tikzcd}$$

Before we prove this, we need some preliminaries.

(1) Recall that $\Theta_n = a_n F^n \otimes E^n \in U \otimes U$ where

$$a_n = (-1)^n q^{-n(n-1)/2} \frac{(q-q^{-1})^n}{[n]!} \in k.$$

One can check that

$$a_n a_m = q^{nm} \begin{bmatrix} n+m \\ n \end{bmatrix} a_{n+m} \quad \forall\, m, n \geq 0 \qquad (*)$$

Consider in $\mathcal{U} \otimes \mathcal{U} \otimes \mathcal{U}$ the elements

$$\Theta'_n = a_n F^n \otimes K^n \otimes E^n, \quad \Theta''_n = a_n F^n \otimes K^{-n} \otimes E^n$$

We claim that

$$(\Delta \otimes 1)\, \Theta_n = \sum_{i=0}^{n} (1 \otimes \Theta_{n-i})\, \Theta''_i$$

This is true since

$$LHS = \sum_{i=0}^{n} a_n q^{i(n-i)} \begin{bmatrix} n \\ i \end{bmatrix} F^i \otimes F^{n-i} K^{-i} \otimes E^n$$

$$RHS = \sum_{i=0}^{n} a_i a_{n-i} (1 \otimes F^{n-i} \otimes E^{n-i})(F^i \otimes K^{-i} \otimes E^i)$$

So equality follows from $(*)$. One shows similarly that

$$(1 \otimes \Delta)\, \Theta_n = \sum_{i=0}^{n} (\Theta_{n-i} \otimes 1)\, \Theta'_i$$

The antiautomorphism $\tau$ satisfies

$$(\tau \otimes \tau)\, \Theta_n = \Theta_n \quad \text{and} \quad (\tau \otimes \tau \otimes \tau)\, \Theta'_n = \Theta''_n$$

Hence we obtain

$$({}^{\tau}\Delta \otimes 1)\, \Theta_n = \sum_{i=0}^{n} \Theta'_i (1 \otimes \Theta_{n-i})$$

$$(1 \otimes {}^{\tau}\Delta)\, \Theta_n = \sum_{i=0}^{n} \Theta''_i (\Theta_{n-i} \otimes 1)$$

(2) $\forall M, N, P \in \mathrm{Rep}(\mathcal{U})^{fd}$, we can construct three automorphims of $M \otimes N \otimes P$:

$$\Theta^f_{12} = \Theta^f \otimes 1, \quad \Theta^f_{23} = 1 \otimes \Theta^f$$

$$\Theta^f_{13} = (1 \otimes P)\, \Theta^f_{12}\, (1 \otimes P)$$

Similarly we have $\Theta_{12}$, $\Theta_{13}$, $\Theta_{23}$ which are defined in the same way just without the $f$.

Also we have $\tilde{f}_{12}, \tilde{f}_{13}, \tilde{f}_{23} \in \mathrm{Aut}(M \otimes N \otimes P)$ in a similar way: for example, $\tilde{f}_{23}$ maps $m \otimes n \otimes p$, with $m \in M_\lambda$, $n \in N_\mu$, $p \in P_\upsilon$, to $f(\mu, \upsilon)\, m \otimes n \otimes p$.

We define operators

$$\Theta' = \sum_{n \geq 0} \Theta'_n \qquad \text{and} \qquad \Theta'' = \sum_{n \geq 0} \Theta''_n$$

We claim that

$$\tilde{f}_{12} \circ \Theta_{13} = \Theta' \circ \tilde{f}_{12} \qquad \text{(i)}$$

$$\tilde{f}_{23} \circ \Theta_{13} = \Theta'' \circ \tilde{f}_{23} \qquad \text{(ii)}$$

$$\tilde{f}_{12} \circ \tilde{f}_{13} \circ (1 \otimes \Theta) = (1 \otimes \Theta) \circ \tilde{f}_{12} \circ \tilde{f}_{23} \qquad \text{(iii)}$$

$$\tilde{f}_{23} \circ \tilde{f}_{13} \circ (\Theta \otimes 1) = (\Theta \otimes 1) \circ \tilde{f}_{23} \circ \tilde{f}_{13} \qquad \text{(iv)}$$

We will only show (i).

For $x = m \otimes n \otimes p \in M_\lambda \otimes N_\mu \otimes P_\upsilon$,

$$\begin{aligned} \mathrm{LHS}(x) &= \sum_{n \geq 0} a_n f(\lambda q^{-2n}, \mu)\, F^n m \otimes n \otimes E^n p \\ &= \sum_{n \geq 0} a_n \mu^n f(\lambda, \mu)\, F^n m \otimes n \otimes E^n p \\ &= \sum_{n \geq 0} a_n f(\lambda, \mu)\, F^n m \otimes K^n m \otimes E^n p \\ &= \Theta' \circ \tilde{f}_{12}(x) = \mathrm{RHS}(x). \end{aligned}$$

<u>Rmk</u>. As a consequence of the above four equations, we can show that

$$\Theta^f_{12} \circ \Theta^f_{13} \circ \Theta^f_{23} = \Theta^f_{23} \circ \Theta^f_{13} \circ \Theta^f_{12}$$

as operators on $M \otimes N \otimes P$, for arbitrary $M, N, P \in \mathrm{Rep}(U)^{fd}$. In particular when $M = N = P$, we get the quantum Yang-Baxter equation.

Now let's get back to proving the main theorem. Recall an additional assumption that

$$\begin{cases} f(\lambda, \mu\nu) = f(\lambda,\mu) f(\lambda,\nu) \\ f(\lambda\mu, \nu) = f(\lambda,\nu) f(\mu,\nu) \end{cases}$$

for all weights $\lambda, \mu, \nu$ of $M, N, P \in \mathrm{Rep}(U)^{fd}$.

We will only show the first diagram

$$\begin{tikzcd}
 & M \otimes (P \otimes N) \arrow[r, phantom, "\cong"] & (M \otimes P) \otimes N \arrow[dr, "R \otimes id"] & \\
M \otimes (N \otimes P) \arrow[ur, "id \otimes R"] \arrow[d, phantom, "\wr\!\!\mid"] & & & (P \otimes M) \otimes N \arrow[d, phantom, "\wr\!\!\mid"] \\
(M \otimes N) \otimes P \arrow[rrr, "R"] & & & P \otimes (M \otimes N)
\end{tikzcd}$$

commutes.

<u>Sketch of proof</u>. The two maps in the upper half are $(\theta \otimes 1) \circ \tilde{f}_{12} \circ P_{12}$ followed by $(1 \otimes \theta) \circ \tilde{f}_{23} \circ P_{23}$.

Obviously, $\begin{cases} P_{12} \circ (1 \otimes \theta) = \theta_{13} \circ P_{12} \\ P_{12} \circ \tilde{f}_{23} = \tilde{f}_{13} \circ P_{12} \end{cases}$

we also have $\tilde{f}_{12} \circ \theta_{13} = \theta' \circ \tilde{f}_{12}$ by (i).

So the upper half can be written as

$$(\theta \otimes 1) \circ \theta' \circ \tilde{f}_{12} \circ \tilde{f}_{23} \circ P_{12} \circ P_{23}$$

The lower half is the composition of a permutation of factors (equal to $P_{12} \circ P_{23}$), a map $\tilde{f}'$ that takes $x \in P_\lambda \otimes (M_\mu \otimes N_\nu)$ to $f(\lambda, \mu\nu) x$, and finally $(1 \otimes \Delta)\theta$. Since $(1 \otimes \Delta)\theta = (\theta \otimes 1)\theta'$, we see that the maps are equal iff

$$\tilde{f}' = \tilde{f}_{12} \circ \tilde{f}_{13}$$

Since RHS takes $x$ to $f(\lambda, \mu) f(\lambda, \nu) x$, we see that the maps are equal iff $f(\lambda, \mu\nu) = f(\lambda, \mu) f(\lambda, \nu)$. ▨

Rmk. If $f$ satisfies the 2 extra conditions for all weights of the form $q^a$ with $a \in \mathbb{Z}$, then

$$f(q^a, q^b) = f(q, q)^{ab} \qquad \forall a, b \in \mathbb{Z}.$$

Further, $f(q,1) f(q,1) = f(q,1) \Rightarrow f(q,1) = 1$

$$f(q,q) f(q,q) = f(q, q^2) = q^{-1} f(q,1) = q^{-1}$$

so $f(q,q)$ is a square root of $q^{-1}$.

Suppose $k$ contains a square root of $q$, denoted by $q^{1/2}$. Then we can define

$$f(q^a, q^b) = (q^{1/2})^{-ab} \qquad \forall a, b \in \mathbb{Z}$$

and then all conditions on $f$ are satisfied for weights of this form.

However, we cannot extend $f$ to all of $\tilde{I}$ this way.

From $f(-1,1) f(-1,1) = f(-1,1)$ we get $f(-1,1) = 1$.

From $f(-1, q^2) = (-1) f(-1,1)$ we get $f(-1, q^2) = -1$

From $f(-1, q^2) = f(-1, q) f(-1, q) = f(1, q)$ we get

$$f(-1, q^2) = 1.$$

We say a $U$-module $M$ is <u>of type 1</u> if all weights have the form $q^a$ with $a \in \mathbb{Z}$. In summary, if $k$ contains a square root of $q$, then we can choose $f$ s.t. we get a commutativity constraint for all f.d. $U$-modules of type 1.