Rep theory of semisimple Lie algebras

Before diving into rep'n theory of f.d. s.s. Lie algebras, let's recall some facts about the Lie algebras themselves.

Let L be a f.d.s.s. Lie algebra (over $\mathbb{C}$ or some alg. closed field k). Then L has a maximal toral subalg. Fix one such algebra $T \subset L$, then

$$L = T \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$$

where $\Phi = \{ \alpha \in T^* \setminus \{0\} : L_\alpha \neq 0 \}$ is the set whose elements are called roots, and

$$L_\alpha = \{ x \in L : [t, x] = \alpha(t) x \ \ \forall t \in T \}$$

Note. $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$.

Some facts about roots.

(1) Let $\langle -, - \rangle$ denote the (dual) Killing form on T^*. $\forall v \in T^* \setminus \{0\}$, let $v^\vee = \dfrac{2v}{\langle v, v \rangle}$, then

$$\langle \mu, v^\vee \rangle = \frac{2 \langle \mu, v \rangle}{\langle v, v \rangle} = \frac{2}{\| v \|^2} \langle \mu, v \rangle$$

The new bracket $\langle -, -^\vee \rangle$ is linear only in the 1st variable and is insensitive to rescaling the inner pdt $\langle -, - \rangle$.

With respect to this new bracket,

$$\forall \alpha, \beta \in \Phi, \ \langle \alpha, \beta^\vee \rangle \in \mathbb{Z}$$

Cartan integers

Furthermore, (·) $k\alpha \cap \Phi = \{\pm\alpha\}$

(·) $\forall \alpha, \beta \in \Phi, \quad \alpha - \langle \alpha, \beta^\vee \rangle \beta \in \Phi$

(2) (sl_2-triples). Let $\alpha \in \Phi$. Then $\dim_k L_\alpha = 1$ and $[L_\alpha, L_{-\alpha}] = k h_\alpha$, where

$$h_\alpha \in T \text{ s.t. } \langle \beta, \alpha^\vee \rangle = \beta(h_\alpha)$$

(Explicitly, $h_\alpha = \frac{2 t_\alpha}{\alpha(t_\alpha)}$, where $t_\alpha \in T \leftrightarrow \alpha \in T^*$)

Moreover,

$s_\alpha = L_{-\alpha} \oplus kh \oplus L_\alpha$ is a Lie subalg of L s.t. $s_\alpha \cong sl_2$, with $h_\alpha \leftrightarrow h \in sl_2$.

(3) There is a finite set $\Delta = \{\alpha_1, \dots, \alpha_n\}$ of roots that forms a basis for $T^* \cong k^n$ and s.t. $\forall \beta \in \Phi$,

$$\beta = \sum_{\alpha \in \Delta} z_\alpha \alpha \quad \text{w/ all } z_\alpha \in \mathbb{Z}_+ \text{ or all } z_\alpha \in -\mathbb{Z}_+.$$

The roots in Δ are called simple roots. They give rise to a partition of Φ as

$$\underbrace{\Phi}_{} = \underbrace{\Phi_+}_{\text{positive roots}} \sqcup \overbrace{\Phi_-}^{\text{negative roots}}$$

There is a partial order on T^* given by

$$\mu \leq \nu \iff \nu - \mu \in \mathbb{Z}_+ \Phi_+ = \mathbb{Z}_+ \Delta = \bigoplus_{i=1}^n \mathbb{Z}_+ \alpha_i$$

With this partial order, we can write

$$\Phi_+ = \{\alpha \in \Phi \mid \alpha > 0\}$$
$$\Phi_- = \{\alpha \in \Phi \mid \alpha < 0\}$$

The root space decomposition takes the form

$$L = N_- \oplus T \oplus N_+ \qquad N_\pm = \bigoplus_{\alpha \in \Phi_\pm} L_\alpha$$

where T is abelian and $N_\pm$ are nilpotent. This is called the triangular decomposition of L. For the classical Lie algs, $N_\pm$ consist of strictly upper/lower triangular matrices.

We will also define the positive and negative Borel subalg to be

$$B_\pm = T \oplus N_\pm .$$

(4) Fix a basis $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ of simple roots of Φ.

Define the root lattice

$$R = \mathbb{Z}\Phi = \sum_{\alpha \in \Phi} \mathbb{Z}\alpha = \bigoplus_{i=1}^{n} \mathbb{Z}\alpha_i \cong \mathbb{Z}^{\oplus n}$$

and the weight lattice

$$\Lambda = \{ \lambda \in T^* \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \ \forall \alpha \in \Phi \}$$

$$= \{ \lambda \in T^* \mid \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z} \ \forall \alpha_i \in \Delta \}$$

not obvious! Exercise 7.2.3 in Lorenz's book

We have $R \subseteq \Lambda$

The weights λ_i s.t. $\langle \lambda_i, \alpha_j^\vee \rangle = \delta_{i,j}$ are called the fundamental weights. They form a $\mathbb{Z}$-basis for Λ, ie.

$$\Lambda = \bigoplus_{i=1}^{n} \mathbb{Z}\lambda_i \cong \mathbb{Z}^{\oplus n}$$

Next, we define

$$\Lambda_+ = \{ \lambda \in T^* \mid \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_+ \ \forall \alpha_i \in \Delta \}$$

$$= \bigoplus_{i=1}^{n} \mathbb{Z}_+ \lambda_i .$$

The weights $\lambda \in \Lambda_+$ are called the dominant weights

THE WEYL GROUP

Let Φ be the set of roots and fix a basis Δ.

For each $\alpha \in \Phi$, let $s_\alpha : T^* \to T^*$

$$s_\alpha(\mu) = \mu - \langle \mu, \alpha^\vee \rangle \alpha$$

Can check that s_α is the reflection through the hyperplane orthogonal to α.

Let $W = \langle s_\alpha \mid \alpha \in \Phi \rangle \subseteq GL(T^*)$

W is called the Weyl group of Φ.

Example (sl_2)

Recall that $sl_2 = kf \oplus kh \oplus ke$

$$f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$[h, e] = 2e \qquad [h, f] = -2f \qquad [e, f] = h$$

kh = a maximal toral subalg

$(kh)^* \cong k$, ie. functionals on kh act as scalars

With this identification, $\Phi = \{\pm 2\}$ since

$$ke = L_2 = \{ x \in sl_2 \mid [h, x] = 2x \}$$

$$kf = L_{-2} = \{ x \in sl_2 \mid [h, x] = -2x \}$$

$$\begin{tikzcd}
0 & L_{-2} = kf \arrow[l, "f"'] \arrow[r, shift left, "e"] \arrow[loop above, "h"] & L_0 = kh \arrow[l, shift left, "f"] \arrow[r, shift left, "e"] \arrow[loop above, "h"] & L_2 = ke \arrow[l, shift left, "f"] \arrow[r, "e"] \arrow[loop above, "h"] & 0
\end{tikzcd}$$

The root lattice is $2\mathbb{Z}$. Choose $\Delta = \{2\}$, so $\Phi_+ = \{2\}$ and $\Phi_- = \{-2\}$. Hence $N_+ = L_2 = ke$

$$N_- = L_{-2} = kf$$

The fund. weight λ satisfies $\langle \lambda, 2^\vee \rangle = 1$

$$\langle \lambda, 2^\vee \rangle = \langle \lambda, \tfrac{2}{4} 2 \rangle = \langle \lambda, 1 \rangle$$

so $\lambda = 1$. The weight lattice is $\mathbb{Z}$. The Weyl group W is generated by reflection s_2 of order 2 so $W \cong \mathbb{Z}/2\mathbb{Z}$, and operates on $T^* = k$ by multiplication by ± 1.

The partial order $\leq$ on $T^* = k$ is given by

$$\mu \leq \lambda \iff \lambda - \mu \in 2\mathbb{Z}_+.$$

REPRESENTATIONS OF SEMISIMPLE LIE ALGEBRAS

Let L be a f.d. s.s. Lie algebra. If $V \in Rep(L)^{fd}$, then we have a decomposition

$$V \cong \bigoplus_{\lambda \in T^*} V_\lambda \qquad (*)$$

where $V_\lambda = \{ x \in V : t \cdot x = \lambda(t) x \ \forall t \in T \}$ is the λ-weight space for V.

Easy check: $L_\alpha \cdot V_\lambda \subseteq V_{\alpha+\lambda}$.

<u>General result</u>. (a) The weights λ occuring in $(*)$ belong to Λ, ie they appear in the weight lattice.

(b) If λ is a weight of V, then the orbit $\mathcal{W}\lambda$ consist of weights of V, all having the same multiplicity:

$$\dim_k V_\lambda = \dim_k V_{w\lambda} \quad \forall\, w \in \mathcal{W}.$$

We will use this to calculate all weights appearing in certain f.d. representations later.

<u>Example</u> ($Rep(sl_2)$)

Let V be a fd irrep of sl_2 of dim'l n, and consider

$$V \equiv \bigoplus_{\lambda \in T^*} V_\lambda \cong \bigoplus_{\lambda \in k} V_\lambda$$

Note $(ke) \cdot V_\lambda = L_2 \cdot V_\lambda \subseteq V_{\lambda+2}$

$(kf) \cdot V_\lambda = L_{-2} \cdot V_\lambda \subseteq V_{\lambda-2}$

In particular, if λ is maximal among all weights wrt $\leq$, and $0 \neq v \in V_\lambda$, then $e \cdot v = 0$ and v generates all of V. Hence the vectors $f^i \cdot v$, $0 \leq i \leq n-1$, forms a basis for V. Using this, we can figure out that the weights are

$$n-1, \quad n-3, \quad \ldots, \quad -(n-3), \quad -(n-1)$$

all occuring with multiplicity 1. A class of representatives of f.d. irreps are given by $\mathrm{Sym}^m(k^2)$ where k^2 is the natural rep'n, $\forall\, m \geq 1$

Note that in this case, the finite dim'l irreps only depend on the maximal weight. This remains true for general f.d.s.s. lie algebras.

Highest weight representations

Let $V \in \mathrm{Rep}(L)^{fd}$. We say that V is a highest weight representation (with highest weight λ) if λ is a maximal weight (wrt $\leq$) of all weights occuring in the weight decomposition $V = \bigoplus_{\mu \in T^*} V_\mu$, and $\exists\, 0 \neq v \in V_\lambda$ s.t. v generates all of V (such a v is called a maximal vector).

Key result. We have bijections

$$\Lambda^+$$

$$\updownarrow$$

$$\{\text{f.d. irreps of } L\} \longleftrightarrow \{\text{f.d. highest weight reps of } L\}$$

A class of representatives of f.d. ireps is given by the Verma modules $V(\lambda)$, $\lambda \in \Lambda_+$.

Some info about $V(\lambda)$:

(1) To compute which weight appears in $V(\lambda)$, first compute all dominant weights $\lesssim \lambda$, then use Weyl group to get the rest. (Lorenz' Proposition 7.16).

(2) The multiplicity $m_\lambda(\mu)$ of a weight μ is given by Kostant's multiplicity formula

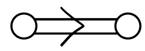
$$m_\lambda(\mu) = \sum_{\sigma \in W} (-1)^{\ell(\sigma)} \underbrace{\nu}_{\text{Kostant's partition function}}(\sigma(\lambda+\rho) - (\mu + \overset{\lambda_1+\dots+\lambda_n}{\rho}))$$

(3) The dimension of $V(\lambda)$ is given by the Weyl dim formula: $\dim(V(\lambda)) = \prod_{\alpha > 0} \frac{\langle \lambda+\rho, \alpha \rangle}{\langle \rho, \alpha \rangle}$.

(4) The multiplicity $M^{\mu}_{\lambda,\gamma}$ of $V(\mu)$ in $V(\lambda) \otimes V(\gamma)$ is given by the Racah-Speicer formula

$$M^{\mu}_{\lambda,\gamma} = \sum_{\sigma \in W} (-1)^{\ell(\sigma)} m_\gamma(\sigma(\mu+\rho) - (\lambda+\rho)).$$

<u>Example</u>. From the Dynkin diagram of B_2



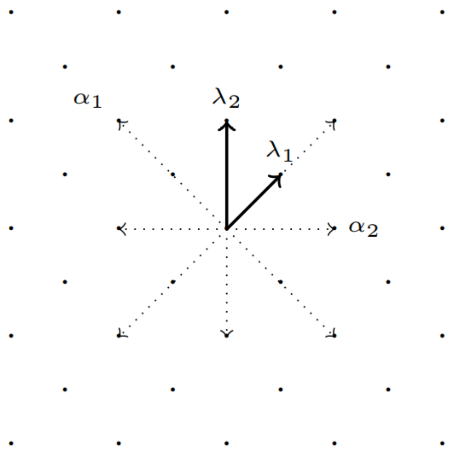
we have 2 simple roots α_1, α_2 w/ $\|\alpha_1\| > \|\alpha_2\|$.

One computes $\text{angle}(\alpha_1, \alpha_2) = 135$ degrees

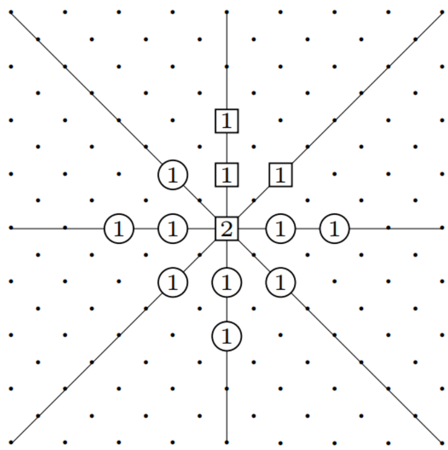
and $\frac{\|\alpha_1\|}{\|\alpha_2\|} = \sqrt{2}$.

and can verify that the fund. weights are

$$\lambda_1 = \frac{\alpha_1}{2} + \alpha_2, \qquad \lambda_2 = \alpha_1 + \alpha_2.$$



(A) Φ and Q



(B) Weight decomposition of $V(2\lambda_2)$

Since $2\lambda_2 \in \Lambda_+$, we can consider $V = V(2\lambda_2)$.

The dominant weights $\preceq 2\lambda_2$ are $2\lambda_2, \lambda_1, \lambda_2, 0$, which are rectangular nodes whose multiplicities inside are computed using (2). The remainder of the weight spaces in circular nodes are computed using the Weyl group.