Ch2 : Representations of $U_q(sl_2)$

Fix a ground field $\mathbb{k}$ and an element $q \in \mathbb{k}$ with $q \neq 0$ and $q^2 \neq 1$.

Recall, the Quantized enveloping algebra $U_q(sl_2)$ is the associative, unital algebra over $\mathbb{k}$ generated by E, F, K, K^{-1} with relations:

(R1) $KK^{-1} = 1 = K^{-1}K$

(R2) $KEK^{-1} = q^2 E$

(R3) $KFK^{-1} = q^{-2} E$

(R4) $EF - FE = \dfrac{K - K^{-1}}{q - q^{-1}} = [K;0]$

GOAL

→ Study finite dim reps of $U_q(sl_2)$ and its center

→ Understand how when:

(i) q is not a root of unity
then rep theory of $U_q(sl_2)$ is similar to rep. theory of $U(sl_2)$. over a field of char 0.

(ii) q is a root of unity
then rep. theory of $U_q(sl_2)$ is similar to the rep theory of $U(sl_2)$ over a field of char p.

Char $\mathbb{k} = 0$ case for $U(sl_2)$

→ $U(sl_2)$ has infinitely many finite dim'l irreducible representations
(Recall $V(m)$ constructed in Pablo's talk)

→ $U(sl_2)$ has infinite dimensional irreducible representation

→ Complete reducibility : every $U(sl_2)$ module is a direct sum of irreducible module. Proof of this involves a central element $c \in Z(U(sl_2))$ called the Casimir element.

Char $\mathbb{k} = p$ case for $U(sl_2)$

→ Every irreducible representation is finite dimensional

→ There are only finitely many irreducible representations.

→ For any $\lambda \neq 0$, $\lambda \in \Bbbk$ and an operator A (think matrix) acting on a vector space V, the $\boxed{\text{eigenspace}}$ corresponding to λ is

$$V_\lambda = \{ v \in V \mid Av = \lambda v \}$$
$$= \{ v \in V \mid (A - \lambda I) v = 0 \}$$

→ The $\boxed{\text{generalized eigenspace}}$ of A for λ is

$$V_{(\lambda)} = \{ v \in V \mid (A - \lambda I)^n v = 0 \text{ for some } n \in \mathbb{N} \}$$

→ When $\Bbbk$ is algebraically closed, any irred polynomial is $f(X) = X - \lambda$ for some λ then

$$M_{(f)} = \{ m \in M \mid (K - \lambda)^n m = 0 \text{ for some } n \in \mathbb{N} \}$$

is a generalized eigenspace

→ When $\Bbbk$ is not algebraically closed

$$f = a_r X^r + a_{r-1} X^{r-1} + \cdots + a_1 X + a_0$$

then $M_{(f)} = \{ m \in M \mid (a_r K^r + \cdots + a_1 K + a_0)^n m = 0 \text{ for some } n \in \mathbb{N} \}$

is a generalized eigenspace.

Prop 2.1 Suppose that q is not a root of unity. Let M be a finite dimensional U-module. There are integers $r, s > 0$ with $E^r M = 0$ and $F^s M = 0$.

Proof: For each irreducible polynomial $f \in \Bbbk[X]$, set

$$M_{(f)} = \{ m \in M \mid f(K)^n m = 0 \text{ for all } n \gg 0 \}$$

Strategy of proof:

Step 1: Show $M = \bigoplus M_{(f)}$

Step 2: $E^r M_{(f)} \subset M_{(f_{-2r})}$

Step 3: For some r, $M_{(f_{-2r})} = 0$

$\therefore$ $E^r M_{(f)} = 0$ for some r

Taking the max $\bar{r}$ of different r, we get $E^{\bar{r}} M = 0$.

DETAILS:

$\boxed{\text{Step 1}}$

(i) $M_{(f)} \cap M_{(g)} \neq 0 \Rightarrow M_{(f)} = M_{(g)}$

Pf: Say $M_{(f)} \cap M_{(g)} \neq 0$
take $m \in M_{(f)} \cap M_{(g)}$
$\Rightarrow \exists\, n \in \mathbb{N}$ s.t. $\boxed{f(K)^n m = 0 = g(K)^n m}$

But f, g are irreducible

$\Rightarrow$ either $f = c\,g$ for $c \in \Bbbk$

OR f, g are relatively prime

If $f \neq cg$ then f^n and g^n are also relatively prime.

Then $\exists$ $a, b \in \Bbbk[X]$ s.t.

$$af^n + bg^n = 1$$

$$\Rightarrow \boxed{(af^n + bg^n)(K)\, m = m}$$

Thus we get a contradiction

(ii) $M_{(f)} = M_{(g)} \iff f = cg$ for some $c \in \Bbbk$, $c \neq 0$

Pf: Easy

(ii) M is a direct sum of distinct $M_{(f)}$.

Pf: by induction on $\dim M$

Step 2

Let $f \in \Bbbk[X]$ be irreducible with $M_{(f)} \neq 0$.
For each $i \in \mathbb{Z}$, set f_i equal to the polynomial

$$f_i(X) = f(q^i X)$$

Consider the automorphism

$$\varphi_i : \Bbbk[X] \longrightarrow \Bbbk[X]$$
$$X \longmapsto q^i X$$

then $\varphi_i(f) = f_i$

since f is irred $\Rightarrow$ $\varphi_i(f) = f_i$ is also irreducible

Using a formula from last talk, we get
$$f(q^{-2}K)\,E = E\,f(K)$$

Applying this r times, we get that
$$f(q^{-2r}K)\,E^r = E^r f(K)$$
i.e. $$f_{(-2r)}(K)\,E^r = E^r f(K)$$

Take $m \in M_{(f)}$, then $f(K)^n m = 0$ for some $n \in \mathbb{N}$

$$\Rightarrow 0 = E^r f(K)^n m = f_{(-2r)}(K)^n E^r m$$

$$\Rightarrow E^r m \in M_{(f_{-2r})}$$

$$\therefore \quad E^r M_{(f)} \subset M_{(f_{-2r})}$$

$\boxed{\text{Step 3}}$

Finally, we will show that $M_{(f_{-2r})} = 0$ for some $r > 0$.

Since f was arbitrary, this will imply that $E^r M_{(f)} = 0$ for some r.

$\rightarrow$ Suppose that $M_{(f_{-2r})} \neq 0 \quad \forall\, r > 0$

Since M is a direct sum of $M_{(g)}$ for different g and M is finite diml.

$\Rightarrow$ $M_{(f_{-2r})} = M_{(f_{-2s})}$ for some $r, s > 0$
$s > r$

then f_{-2r} and f_{-2s} have to be proportional

But they have the same constant term
(recall $f_{-2r} = f(q^{-2r}x)$)

$\therefore$ $f_{-2r} = f_{-2s}$

but, if f has degree n, then the leading coefficients differ by a factor of $q^{2(s-r)n}$.

Since q is not a root of unity, this factor is not 1.

$\therefore$ f_{-2s} and f_{-2r} can't be equal
$\Rightarrow\Leftarrow$

Hence, $E^r M(f) = 0$ for some r
Taking the max of all r for different f, we get $E^r M = 0$.

$\blacksquare$

DEFINITIONS

If M is a U-module, then set for $\lambda \in \Bbbk$, $\lambda \neq 0$
$$M_\lambda = \{ m \in M \mid Km = \lambda m \}$$
i.e. M_λ is the eigenspace of K acting on M for the eigenvalue λ.

→ We call M_λ a weight space of M.
→ The λ with $M_\lambda \neq 0$ are called the weights of M.

REMARKS :

→ For $\lambda \neq \lambda'$, $M_\lambda \cap M_{\lambda'} = 0$
→ $E M_\lambda \subset M_{q^2\lambda}$ and $F M_\lambda \subset M_{q^{-2}\lambda}$
→ This shows that the sum of M_λ is a submodule of M.
More precisely, for any λ, $\bigoplus_{n\in\mathbb{Z}} M_{q^{2n}\lambda}$ is a submodule

→ If M is simple and $M_\lambda \neq 0$, then
$$M = \bigoplus_{n \in \mathbb{Z}} M_{q^{2n}\lambda}$$
(if q is not a root of unity this sum runs over all integers, otherwise over a finite set)

→ While in general we might not have λ such that $M_\lambda \neq 0$, if $\Bbbk$ is alg. closed and M is finite diml. then K has a nonzero eigenspace i.e. $\exists\ \lambda \neq 0$ s.t. $M_\lambda \neq 0$.

Prop 2.3 Suppose that q is not a root of unity and that $\mathrm{char}(\mathbb{k}) \neq 2$. Let M be a finite dimensional U-module. Then M is a direct sum of its weight spaces. All weights of M have the form $\pm q^a$ with $a \in \mathbb{Z}$.

Proof: An endomorphism of a f.d. vector space is diagonalizable if and only if its minimal polynomial splits into linear factors.

The eigenvalues are then the roots of the minimal polynomial.

We will show the minimal polynomial of the endomorphism K acting on M has the form $\prod_i (X-\lambda_i)$ where λ_i are distinct elements of the form $\pm q^a$.

By Prop 2.1, $\exists\ s \in \mathbb{Z},\ s>0$ s.t.
$F^s M = 0$

Set $h_r = \prod_{j=-(r-1)}^{r-1} [K; r-s+j]$ for all integers $r \geq 0$

$\left(\text{recall } [K;a] = \dfrac{Kq^a - K^{-1}q^{-a}}{q-q^{-1}} \right)$

$\rightarrow\ h_0 = 1$

By induction on r, $0 \le r \le s$, one can check that

$$\alpha = F^{s-r} h_r M = 0$$

$\underline{r=0}$ $\alpha = F^s M = 0$

$\underline{r=s}$ tedious calculation get that

$$0 = h_s M = \Big(\prod_{j=-(s-1)}^{s-1} \underbrace{(q-q^{-1})^{-1} q^j}_{\text{remove}} \underbrace{K^{-1}}_{\text{multiply by appropriate power of } K} (K^2 - q^{-2j}) \Big) M$$

to get

$$\left(\prod_{j=-(s-1)}^{(s-1)} (K^2 - q^{-2j}) \right) M = \left(\prod_{j=-(s-1)}^{(s-1)} (K - q^{-j})(K + q^{-j}) \right) M = 0$$

Since the minimal polynomial f of K satisfies

$$f(K)\, M = 0$$

$\Rightarrow$ $f(X)$ will divide $\displaystyle\prod_{j=-(s-1)}^{(s-1)} (X - q^{-1})(X + q^{-1})$

$\Rightarrow$ f splits into distinct factors with each occuring with multiplicity 1.

$\therefore$ All weights are of the form $\pm q^a$ and all weight spaces are one dimensional.

For each $\lambda \in \mathbb{k}$, $\lambda \neq 0$ there is an infinite diml. U-module $M(\lambda)$ with basis $m_0, m_1, m_2, \ldots$ such that for all i

$$\left.\begin{aligned} K m_i &= \lambda q^{-2i} m_i, \\ F m_i &= m_{i+1} \\ E m_i &= \begin{cases} 0 & , \text{ if } i=0 \\ [i] \dfrac{\lambda q^{1-i} - \lambda^{-1} q^{i-1}}{q-q^{-1}} m_{i-1} & , \text{ otherwise} \end{cases} \end{aligned}\right] \circledast$$

Define $$M(\lambda) = \frac{U}{(UE + U(K-\lambda))} \qquad \text{--- (1)}$$

with $m_i =$ coset of F^i

→ Then the formula $\circledast$ follow from definitions of U.

→ The linear independence of m_i follows from the fact that
$\{ F^s K^n E^r \mid s, n, r \in \mathbb{Z},\ s, r \geq 0 \}$ form as basis of U.

→ It is clear from (1) that $M(\lambda)$ satisfies the following universal property:

If M is a U-module and $m \in M$ a vector with $Em = 0$ and $Km = \lambda m$, then there is a unique homomorphism of U-modules $\varphi : M(\lambda) \longrightarrow M$ with
$\varphi(m_0) = m$