Representations of $U_q(sl_2)$ continued ....

Fix a ground field $\Bbbk$ and an element $q \in \Bbbk$ with $q \neq 0$ and $q^2 \neq 1$.

Recall, the Quantized enveloping algebra $U_q(sl_2)$ is the associative, unital algebra over $\Bbbk$ generated by E, F, K, K^{-1} with relations:

(R1) $KK^{-1} = 1 = K^{-1}K$
(R2) $KEK^{-1} = q^2 E$
(R3) $KFK^{-1} = q^{-2} E$
(R4) $EF - FE = \dfrac{K - K^{-1}}{q - q^{-1}} = [K; 0]$

→ Last time we saw the following:

Prop 2.1 Suppose that q is not a root of unity. Let M be a finite dimensional U-module. There are integers $r, s > 0$ with $E^r M = 0$ and $F^s M = 0$.

For a U-module M, we defined
→ Weight spaces for $\lambda = \Bbbk$, $\lambda \neq 0$
$$M_\lambda = \{ m \in M \mid Km = \lambda m \}$$
λ is the called the weight of M_λ.

Prop 2.3 Suppose that q is not a root of unity and that $\mathrm{char}(\Bbbk) \neq 2$. Let M be a finite dimensional U-module. Then M is a direct sum of its weight spaces. All weights of M have the $\pm q^a$ with $a \in \mathbb{Z}$.

Last time we had defined certain infinite dimensional U-modules as following

Take $\lambda \in \mathbb{k}$, $\lambda \neq 0$ and define
$$M(\lambda) = \frac{U}{(UE + U(K-\lambda))}$$

→ $M(\lambda)$ has a basis $\{m_i\}_{i \in \mathbb{Z}, i \geq 0}$ where
$$m_i = \text{coset of } F^i$$

→ The action of U on basis elements is given as follows:

$$\circledast \quad \boxed{\begin{aligned} K m_i &= \lambda q^{-2i} m_i \\ F m_i &= m_{i+1} \\ E m_i &= \begin{cases} 0 & i=0 \\ [i] \dfrac{\lambda q^{1-i} - \lambda^{-1} q^{i-1}}{q-q^{-1}} m_{i-1} & \text{otherwise} \end{cases} \end{aligned}}$$

→ $M(\lambda)$ satisfies the following universal property:
If M is a U-module and $m \in M$ a vector with $Em = 0$ and $Km = \lambda m$, there exists a unique homomorphism of U-modules
$$\varphi : M(\lambda) \longrightarrow M$$
with $\varphi(m_0) = m$.

Remarks:

(1) If $\lambda = \pm q^a$ for some $a \in \mathbb{Z}$, then the action of E on m_i can be simplified as follows:

$$E m_i = \pm [i][a+1-i]\, m_i \quad \text{for } \lambda = \pm q^a$$

(2) Since $K m_i = \lambda q^{-2i} m_i$, $m_i \in M(\lambda)_{\lambda q^{-2i}}$

→ If q is not a root of unity, then $q^{-2i}\lambda$ are distinct and we get

$$M(\lambda)_{q^{-2i}\lambda} = \Bbbk m_i \quad \text{for all } i \geq 0$$

→ If q is a primitive l-th root of unity, then $q^{-2i}\lambda = q^{-2j}\lambda$ if and only if l divides $2(j-i)$.

So there are 2 cases:

l odd: then weight spaces in $M(\lambda)$ are

$$M(\lambda)_{q^{-2i}\lambda} = \bigoplus_{n \geq 0} \Bbbk m_{i+nl}$$

$l = 2l'$ is even: then the weight spaces are

$$M(\lambda)_{q^{-2i}\lambda} = \bigoplus_{n \geq 0} \Bbbk m_{i+nl'}$$

Description of simple U-modules

Prop 2.5: Suppose that q is not a root of unity. Let $\lambda \in \mathbb{k}$, $\lambda \neq 0$.

(i) If $\lambda \neq \pm q^n$ for all integers $n \geq 0$, then the U-module $M(\lambda)$ is simple.

(ii) If $\lambda = \pm q^n$ for some integer $n \geq 0$, then the $\{m_i \mid i \geq n+1\}$ span a submodule of $M(\lambda)$ isomorphic to $M(q^{-2(n+1)}\lambda)$; this is the only submodule of $M(\lambda)$ different from 0 and $M(\lambda)$.

PROOF

→ Take a nonzero submodule $M' \subset M(\lambda)$.

→ Since M' is K-stable, it is a direct sum of its weight spaces, i.e., of all $M' \cap M(\lambda)_\mu$.

→ But $M(\lambda)_{q^{-2i}\lambda} = \mathbb{k} m_i$

$\Rightarrow M' = \bigoplus_{i \in I} \mathbb{k} m_i$ for some m_i

Let j be the minimal element in I.

then $m_i = F^{i-j} m_j \in M'$ $\forall\ i \geq j$

$\therefore M' = \text{span}\{m_i \mid i \geq j\}$

→ If $j=0$ then $M' = M(\lambda)$

→ So let's assume $j > 0$

But $E m_j = c\, m_{j-1}$ for some $c \in \mathbb{k}$

and j is least index in I

$\Rightarrow E m_j = 0$

$$\Rightarrow \quad c = [j] \frac{\lambda q^{1-j} - \lambda^{-1} q^{j-1}}{q - q^{-1}} = 0$$

$$\Rightarrow \quad \lambda q^{1-j} = \lambda^{-1} q^{j-1}$$

$$\Rightarrow \quad \lambda^2 = q^{2(j-1)}$$

$$\Rightarrow \quad \lambda = \pm q^{j-1}$$

Thus, if $\lambda \neq \pm q^n$ then $Em_j \neq 0$, hence $M' = M(\lambda)$

$\therefore$ $M(\lambda)$ is simple.

If $\lambda = \pm q^n$ then there is at most one submodule of $M(\lambda)$ different from 0 and $M(\lambda)$.

However if $\lambda = \pm q^n$ then

$E m_{n+1} = 0$ & $K m_{n+1} = \lambda q^{-2(n+1)} m_{n+1}$

Set $M' = \text{span}\{m_i \mid i \geq n+1\}$

Then by the universal property of U, $\exists$ a morphism of U-modules

$$M(\lambda q^{-2(n+1)}) \longrightarrow M'$$

sending $m_0 \longmapsto m_{n+1}$

But $M(\lambda q^{-2(n+1)}) = M(\pm q^n q^{-2n-2}) = M(\pm q^{-n-2})$

$\blacksquare$

Theorem 2.6: Suppose that q is not a root of unity. There are for each integer $n \geq 0$ there are two simple U-modules

(i) $L(n,+)$ with basis $m_0, m_1, \dots, m_n$ and action

$$K \cdot m_i = q^{n-2i} m_i$$

$$F \cdot m_i = \begin{cases} m_{i+1} & \text{if } i < n \\ 0 & \text{if } i = n \end{cases}$$

$$E \cdot m_i = \begin{cases} [i][n+1-i] m_{i-1} & \text{if } i > 0 \\ 0 & i = 0 \end{cases}$$

(ii) $L(n,-)$ with basis $m_0', m_1', \dots, m_n'$ and action

$$K \cdot m_i' = -q^{n-2i} m_i'$$

$$F \cdot m_i' = \begin{cases} m_{i+1}' & i < n \\ 0 & i = n \end{cases}$$

$$E \cdot m_i' = \begin{cases} -[i][n+1-i] m_{i-1}' & i > 0 \\ 0 & i = 0 \end{cases}$$

for all $0 \leq i \leq n$

Each simple U-module of dimension $n+1$ is isomorphic to $L(n,+)$ or $L(n,-)$.

PROOF: Define $L(n, \pm) = \dfrac{M(\pm q^n)}{M'}$

where $M' = \text{span}\{m_i \mid i \geq n+1\} \subset M(\pm q^n)$ then the simplicity of $L(n, \pm)$ follows from Prop. 2.5.

Take m_i (resp m_i') to be the images of $m_i \in M(\pm q^n)$ under the quotient map.
Then the actions of E, F, K on m_i, m_i' follows from definition.

Eg: $E \cdot m_i = [i][n+1-i] m_{i-1}$ for $m_i \in M(\pm q^n)$
After taking quotient by M', get above relation.

Let M be any finite dimensional simple U-module

→ By Prop 2.3, $M = \bigoplus_{\lambda} M_\lambda$

→ Since $\dim M < \infty$, the set of λ with $M_\lambda \neq 0$ is finite.

→ Then we can find λ s.t. $M_\lambda \neq 0$ but $M_{q^2\lambda} = 0$

→ Pick $m \in M_\lambda$, $m \neq 0$. We had seen earlier that $Em = M_{q^2\lambda}$

$\Rightarrow \quad Em = 0$

→ By universal property of U-module $M(\lambda)$, we get a homo.

$\varphi : M(\lambda) \longrightarrow M$

→ Since M is simple, φ is surjective

→ If $\lambda \neq \pm q^n$ then $M(\lambda)$ is simple. Since φ is non-zero, by Schur's lemma, it has to be an iso. But M is finite dim.

$\Rightarrow \quad \lambda = \pm q^n$ for some $n \in \mathbb{Z}$, $n \geq 0$.

$$\therefore \quad M \cong \frac{M(\lambda)}{\ker \varphi}$$

$\ker \varphi$ is a submodule of $M(\lambda)$

→ But only possible submodule is M' (by Prop 2.4)

→ $\therefore \quad M \cong \dfrac{M(\lambda)}{M'} = L(n, \pm)$.

■

Remark: If $\Bbbk$ has char 2, then
$$M(+q^n) \cong M(-q^n)$$
because
$$2q^n = q^n + q^n = 0$$
$$\Rightarrow q^n = -q^n$$

This in turn implies $L(n,+) \cong L(n,-)$

Complete reducibility of $U_q(\mathfrak{sl}_2)$

We start by analyzing the center of $U_q(\mathfrak{sl}_2)$.

Set
$$C = FE + \frac{Kq + K^{-1}q^{-1}}{(q-q^{-1})^2}$$

Using (R4), we can write

$$C = EF + \frac{K-K^{-1}}{q-q^{-1}} + \frac{Kq + K^{-1}q^{-1}}{(q-q^{-1})^2}$$
$$= EF + \frac{Kq + K^{-1}q^{-1} + (K-K^{-1})(q-q^{-1})}{(q-q^{-1})^2}$$

$$\therefore \quad C = EF + \frac{Kq^{-1} + K^{-1}q}{(q-q^{-1})^2}$$

Lemma 2.7: a) The element C is central in U.
b) C acts on each $M(\lambda)$ as scalar multiplication by $\frac{(\lambda q + \lambda^{-1} q^{-1})}{(q-q^{-1})^2}$

c) C acts on $M(\lambda)$ and $M(\mu)$ by the same scalar if and only if $\lambda = \mu$ or $\lambda = \mu^{-1} q^{-2}$.

PROOF: Recall that U is a graded ring with $\deg(E) = 1$, $\deg(K) = 0$, $\deg(F) = -1$

a) Follows from elementary calculations & some tricks developed in Pablo's talk.

b)
$$C m_0 = EF m_0 + \frac{Kq + K^{-1}q^{-1}}{(q-q^{-1})^2} m_0$$
$$= (E m_1) + \frac{q(K m_0) + q^{-1}(K^{-1} m_0)}{(q-q^{-1})^2}$$
$$= [1]\left(\frac{\lambda q^{1-1} - \lambda^{-1} q^{1-1}}{q - q^{-1}}\right) m_0 + \left(\frac{q\lambda + q^{-1}\lambda^{-1}}{(q-q^{-1})^2}\right) m_0$$
$$= \left\{\frac{(\lambda - \lambda^{-1})(q-q^{-1}) + q\lambda + q^{-1}\lambda^{-1}}{(q-q^{-1})^2}\right\} m_0$$
$$= \left[\frac{\lambda q + \lambda^{-1} q^{-1}}{(q-q^{-1})^2}\right] m_0$$

(c) Using (b) we get the same scalar if and only if
$$\lambda q + \lambda^{-1} q^{-1} = \mu q + \mu^{-1} q^{-1}$$
$$\Leftrightarrow (\lambda - \mu) q = (\mu^{-1} - \lambda^{-1}) q^{-1}$$
$$= \frac{(\lambda - \mu)}{\lambda\mu} q^{-1}$$
$$\Leftrightarrow \lambda = \mu \quad \text{or} \quad q = \lambda^{-1} \mu^{-1} q^{-1}$$

Remark: If $\varphi : M(\lambda) \to M$ is a homo. then C also acts on $\varphi(M(\lambda)) \subset M$ as scalars.

Lemma: Suppose that q is not a root of unity. Let L and L' be finite dim'l simple U modules. If C acts on L and L' by the same scalar then $L \cong L'$.

PROOF: By theorem 2.6, $\exists$ integers $n, m \geq 0$ and signs $\varepsilon, \varepsilon' \in \{1, -1\}$ such that

$$L = L(n, \varepsilon) = \frac{M(\varepsilon q^n)}{M'} \quad \text{and} \quad L' = L(m, \varepsilon') = \frac{M(\varepsilon' q^m)}{M'}$$

If C acts by the same scalar on L and L', then it also acts by the same scalar on $M(\varepsilon q^n)$ & $M(\varepsilon' q^m)$.

Then using last lemma, we get that

$\varepsilon q^n = \varepsilon' q^m$ ($\Rightarrow L \cong L'$ as desired) or $\varepsilon q^n = \varepsilon' q^2 q^{-m}$

$$\Rightarrow q^{n+m+2} = \varepsilon \varepsilon' = \pm 1$$

$\Rightarrow \Leftarrow$ since q is not a root of unity.

$\blacksquare$

Theorem 2.9 Suppose that q is not a root of unity. Let M be a finite dimensional U-module that is a direct sum of its weight spaces. Then M is a semisimple U-module.

(note that when $\mathrm{char}(\Bbbk) \neq 2$, then M being finite diml
$\Rightarrow$ M = direct sum of weight spaces
by Prop 2.3)

PROOF: Let M be a finite diml. U-module.

$\rightarrow$ Pick a composition series
$0 = M_0 \subset M_1 \subset M_2 \cdots \subset M_r = M$ of M

$\rightarrow$ Each $\frac{M_i}{M_{i-1}}$ is simple, thus iso. to $L(n, \pm)$ for some $n \geq 0$

$\rightarrow$ So C acts a scalar, say μ_i, on M_i/M_{i-1}

$\rightarrow$ Then $\prod_{i=1}^{r} (C - \mu_i)$ annihilates M

$\therefore$ the minimal poly. has to divide $\prod_{i=1}^{r} (C - \mu_i)$

$\Rightarrow$ the minimal poly. of C acting on M splits into linear factors, and M is direct sum of the generalized eigenspaces of C

$M = \bigoplus_{\mu} M_{(\mu)}$ where
$M_{(\mu)} = \{ m \in M \mid (C-\mu)^k m = 0 \text{ for some } k > 0 \}$

→ Since C is central in U, $\forall$ $u \in U$ $m \in M_{(\mu)}$, $(C-\mu)^k um = u(C-\mu)^k m$

$\therefore$ $um \in M_{(\mu)}$

$M_{(\mu)}$ is a submodule of M.

$\therefore$ It suffices to prove that each $M_{(\mu)}$ is semisimple.

Let us assume that $M = M_{(\mu)}$ for some μ.

→ Then $C-\mu$ acts nilpotently on M, hence on each M_i / M_{i-1}.

→ On the other hand, C acts as mult. by μ_i on M_i/M_{i-1}.

$\Rightarrow$ $\mu_i = \mu$ for all i

→ By lemma 2.8, $\exists$ $n \geq 0$ and a sign ε such that each M_i/M_{i-1} is isomorphic to $L(n, \varepsilon)$.

→ By assumption, M is a direct sum of its weight spaces

$$M = \oplus_\nu M_\nu$$

→ If N is a submodule of M, then $N = \oplus_\nu N_\nu$ and $N_\nu = N \cap M_\nu$ for all ν.

$$\Rightarrow \dim M_\nu = \dim N_\nu + \dim \left(\frac{M}{N}\right)_\nu$$

→ Applying this repeatedly to the composition series, we get that
$\dim(M_\nu) = \sum_{i=1}^{r} \dim\left(\frac{M_i}{M_{i-1}}\right)_\nu = r \dim L(n,\varepsilon)_\nu$

→ When $\nu = \varepsilon q^n$ then we had seen before in thm 2.6 that
$L(n,\varepsilon)_\nu = \mathbb{k} m_0$
and $L(n,\varepsilon)_{q^2\nu} = 0$

$\therefore$ $\dim M_\nu = r \dim L(n,\varepsilon)_\nu = r$ for $\nu = \varepsilon q^r$
and
$\dim M_{q^2\nu} = 0 \Rightarrow M_{q^2\nu} = 0$

→ $\therefore$ For any $v \in M_\nu$ we have therefore $Ev = 0$

So the submodule Uv is the homomorphic image of $M(\nu)$

→ Since Uv is finite dim^l
$\Rightarrow Uv \cong L(n,\varepsilon)$

→ Choose a basis $v_1, \dots, v_r$ of M_ν.

Then $M = \sum_{i=1}^{r} Uv_i$, since
$\left(M/\sum_{i=1}^{r} Uv_i\right)_\nu = 0$

& since each composition factor L of $M/\left(\sum_{i=1}^{r} Uv_i\right)$ is isomorphic to $L(n,\varepsilon)$.

satisfies $L_\nu \neq 0$.

Since $\dim M = r \dim L(n, \varepsilon)$

$$= \sum_{i=1}^{r} \dim(Uv_i)$$

$$\therefore \quad M = \bigoplus Uv_i$$

Hence M is semisimple.