LIE ALGEBRAS

§ 1.1

- A Lie algebra is a complex vector space $\mathfrak{g}$ equipped with an antisymmetric bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ (bracket) which satisfies the Jacobi identity
$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$$

<u>Example</u>: $\mathfrak{g} = \mathfrak{gl}(V) = \mathrm{End}(V)$
with $[f,g] = fg - gf \quad \forall\, g,f \in \mathfrak{gl}(V)$

- An ideal in a Lie algebra $\mathfrak{g}$ is a vector subspace $\mathfrak{h} \subset \mathfrak{g}$ such that $[x,y] \in \mathfrak{h}$ for all $x \in \mathfrak{g}$ and $y \in \mathfrak{h}$

- Lie algebras with $[x,y] = 0 \;\; \forall\, x,y$ are called abelian. Nonabelian Lie algebras without proper nontrivial ideals are known as simple.

<u>Example</u>: One can decompose $\mathfrak{gl}_n = \mathbb{C} \oplus \mathfrak{g}$ into a direct sum of ideals, where
$\mathbb{C}$ = scalar multiples of I_n
$\mathfrak{g}$ = traceless matrices $= \mathfrak{sl}_n$

§ 1.2

Semisimple Lie algebras:

- Each Lie algebra $\mathfrak{g}$ naturally acts on itself via the Lie algebra homomorphism

$$\begin{aligned} \mathrm{ad}_{\mathfrak{g}} : \mathfrak{g} &\longrightarrow \mathfrak{gl}(\mathfrak{g}) \\ x &\longmapsto \{ y \mapsto [x,y] \} \end{aligned}$$

  This is called the adjoint representation.

- The Killing form of $\mathfrak{g}$ is the invariant complex symmetric bilinear form

$$K(x,y) = \mathrm{Tr}(\mathrm{ad}_{\mathfrak{g}}(x)\, \mathrm{ad}_{\mathfrak{g}}(y)) \quad \forall\, x,y \in \mathfrak{g}$$

- A Lie algebra is semisimple if its Killing form is nondegenerate.

- An element $x \in \mathfrak{g}$ satisfying that $\mathrm{ad}_{\mathfrak{g}}(x)$ is a diagonalizable endomorphism is called a semisimple element.
  Being a semisimple algebra does not imply that all $x \in \mathfrak{g}$ are semisimple

- Any subalgebra $\mathfrak{g} \subset \mathfrak{gl}_n$ generated by semisimple elements is known as a toral subalgebra. Such a subalgebra is abelian and simultaneously diagonalizable.

§ 1.3 Root space decomposition

- Let $\mathfrak{t}$ denote the maximal toral subalgebra (also called as Cartan subalgebra) of $\mathfrak{g}$. The adjoint action of $\mathfrak{t}$ on $\mathfrak{g}$ allows us to decompose a semisimple Lie algebra as

$$\mathfrak{g} \cong \mathfrak{t} \oplus \bigoplus_{\alpha \in \mathfrak{t}^* \setminus \{0\}} \mathfrak{g}_\alpha$$

a direct sum of vector spaces, where for each $\alpha \in \mathfrak{t}^* = \mathrm{Hom}(\mathfrak{t}, \mathbb{C})$,

$$\mathfrak{g}_\alpha := \{x \in \mathfrak{g} : [t, x] = \alpha(t) x \quad \forall \; t \in \mathfrak{t}\}$$

The $\mathfrak{g}_\alpha$ are called root spaces

- The root system of $\mathfrak{g}$, denoted Φ, is the collection of all non-zero functionals $\alpha \in \mathfrak{t}^*$ such that $\mathfrak{g}_\alpha \neq 0$.

- FACTS : $\rightarrow$ $\mathfrak{t} = \mathfrak{g}_0$
  $\rightarrow$ Killing form restricted to $\mathfrak{t}$ is nondegenerate.

- Using the Killing form on $\mathfrak{t}$, $\mathfrak{t}^*$ becomes a Euclidean space.

- If $\mathfrak{g} \subset \mathfrak{gl}_n$ with $h_1, h_2, \ldots, h_r$ a basis for $\mathfrak{t}$, we can think of these as diagonal matrices. Then the functionals $\varepsilon_1, \ldots, \varepsilon_n$ with $\varepsilon_i(h_j) =$ i^{th} diagonal entry of h_j, form a spanning set for $\mathfrak{t}^*$.

- A root system Φ is called irreducible if we can partition $\Phi = \Phi_1 \cup \Phi_2$ such that $\langle \alpha, \beta \rangle = 0$ $\forall$ $\alpha \in \Phi_1$ and $\beta \in \Phi_2$.

- For all simple Lie algebras $\mathfrak{g}$, the root system Φ is irreducible. There is a basis of t^* called simple roots such that every root is either a sum of simple roots with nonnegative coefficients (positive roots), or a sum of simple roots with nonnegative coefficients (negative roots). The set of simple roots is denoted as Δ.

- $P := \mathbb{Z}\Phi$ is called the root lattice.

- P has a partial dominance ordering for which $\alpha \prec \beta$ if and only if $\beta - \alpha$ is a sum of positive roots.

EXAMPLE : ($\mathfrak{sl}_3$)

Let E_{ij} = matrix with entry 1 in $(i,j)^{th}$ pos.

When $i<j$, set $e_{ij} := E_{ij}$

$i>j$ $\quad f_{ij} := E_{ij}$

$h_i = E_{ii} - E_{i+1\, i+1}$

then $\mathfrak{sl}_3$ has basis

$$\{e_{12}, e_{23}, e_{13}, h_1, h_2, f_{32}, f_{21}, f_{31}\}$$

→ h_1, h_2 generate the maximal toral subalgebra of $\mathfrak{t} \subset \mathfrak{sl}_3$.

→ The functional $\varepsilon_1, \varepsilon_2, \varepsilon_3$ span $\mathfrak{t}^*$ but satisfy $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0$ since $(\varepsilon_1+\varepsilon_2+\varepsilon_3)(M) = tr(M) = 0 \quad \forall \; M \in \mathfrak{sl}_3$

→ Simple roots are $\alpha_1 := \varepsilon_1 - \varepsilon_2$, $\alpha_2 := \varepsilon_2 - \varepsilon_3$

→ For any semisimple Lie alg, either $\mathfrak{g}_\alpha = 0$ or $\dim(\mathfrak{g}_\alpha) = 1$.

For $\mathfrak{g} = \mathfrak{sl}_3$ we get the decomposition

$$\mathfrak{g} \cong \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_1+\alpha_2} \oplus \mathfrak{t} \oplus \mathfrak{g}_{-\alpha_1-\alpha_2} \oplus \mathfrak{g}_{-\alpha_2} \oplus \mathfrak{g}_{-\alpha_1}$$

here $\mathfrak{g}_{\alpha_1} = span(e_{12})$

$\mathfrak{g}_{-\alpha_2} = span(f_{32})$ ... so on

Thus

$$\Phi = \{\alpha_1, \alpha_2, \alpha_1+\alpha_2, -\alpha_1-\alpha_2, -\alpha_2, -\alpha_1\}$$

$$\Delta = \{\alpha_1, \alpha_2\}$$

The angle between α_1 and α_2 is given by

$$K^*(\alpha_1,\alpha_2) = K(\alpha_1^*, \alpha_2^*) = K(h_1, h_2)$$

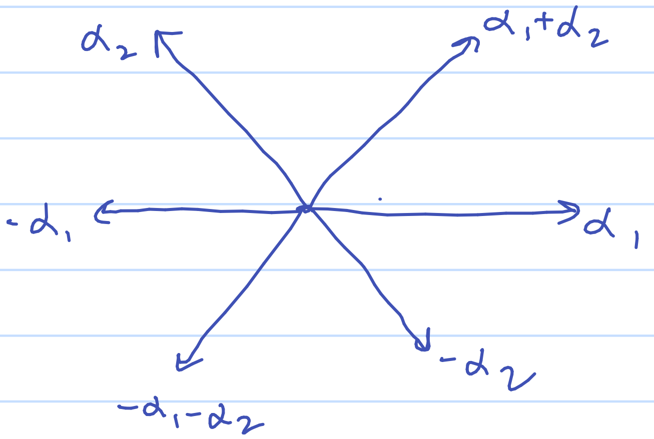
$$= Tr(\, ad(h_1)\, ad(h_2)) = -6$$

But $\|\alpha_1\| = \sqrt{Tr(\, ad(h_1)\, ad(h_1))} = 2\sqrt{3}$

$\therefore$ angle b/w α_1, α_2 $= \arccos\left(\dfrac{-6}{2\sqrt{3}\; 2\sqrt{3}}\right)$

$$= \arccos\left(\frac{-1}{2}\right) = \frac{\pi}{3}.$$

This gives the following geometric realization



§ 1.4 The classification theorem

- Say $\mathfrak{g}$ is a simple Lie algebra. It induces the Killing form on $\mathfrak{g}$ which is symmetric bilinear nondeg.

$$K : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{k} \quad \rightsquigarrow \quad \theta : \mathfrak{g} \longrightarrow \mathfrak{g}^*$$

K nondeg $\Rightarrow$ θ is surjective

$\mathfrak{g}$ simple $\Rightarrow$ by Schur's lemma θ is an iso and determined uniquely upto a scalar.

$\therefore$ K is determined uniquely upto a scalar

$\Rightarrow$ The induced nondeg. bilinear form on $\mathfrak{t}^*$ is unique upto a scalar.

Set $\langle \cdot, \cdot \rangle$ to be this form normalized so that the shortest root has squared length 2.

- Using $\langle \cdot, \cdot \rangle$ on $\mathfrak{t}^*$, we can define

$\boxed{\text{Cartan matrix}}$ If $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ then the cartan matrix has entries

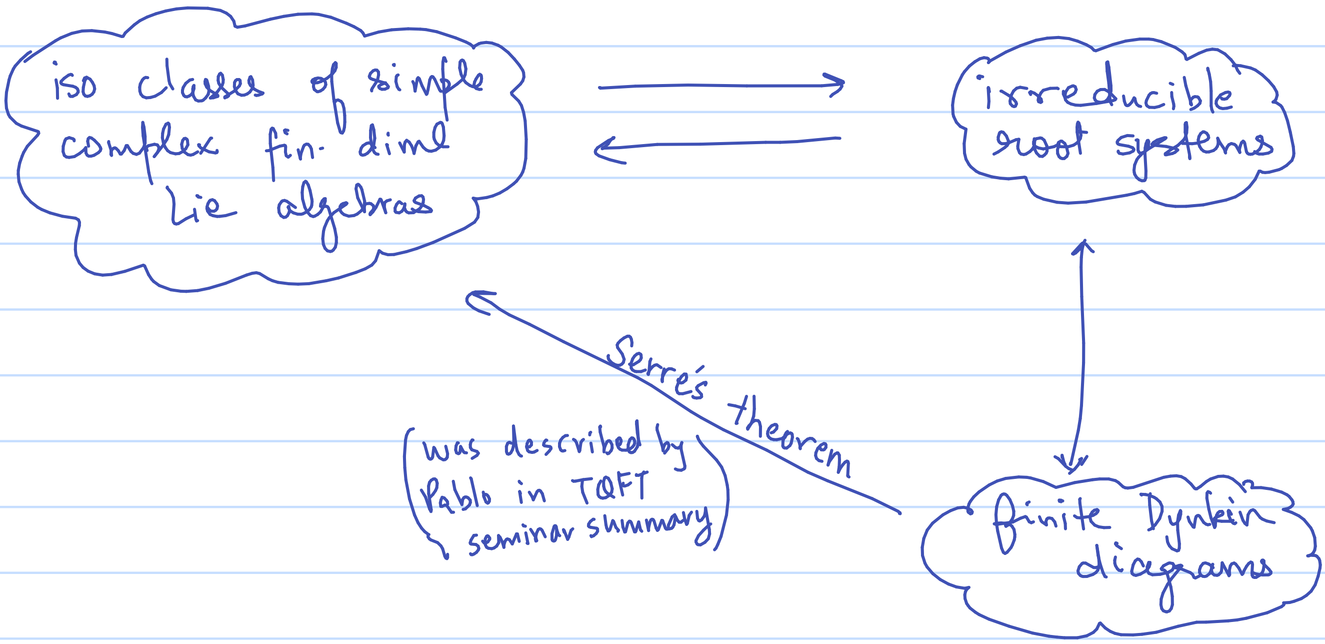
$$c_{ij} := \langle \alpha_i, \alpha_j^\vee \rangle \quad \text{where} \quad \underbrace{\alpha_j^\vee}_{\text{coroot of } \alpha_j} = \frac{2\alpha_j}{\|\alpha_j\|^2}$$

Dynkin Diagram It is a connected graph with

- r vertices
- $c_{ij}\,c_{ji}$ edges b/w vertices i and j
- an arrow on each multiple edge pointing towards the shorter root

- If all the roots have the same length the Dynkin diagram, irreducible root system, and the corresponding simple Lie algs are referred to as simply-laced.

UPSHOT:



finite Dynkin digrams