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A brief introduction to Lie algebras.

Definition:

An algebra over k is a k-vector space A with a multiplication $m: A \times A \longrightarrow A$ satisfying:

(1) $m(v, w_1 + w_2) = m(v, w_1) + m(v, w_2)$,

(2) $m(v_1 + v_2, w) = m(v_1, w) + m(v_2, w)$,

(3) $\lambda m(v, w) = m(\lambda v, w) = m(v, \lambda w)$, for all $v, v_1, v_2, w, w_1, w_2 \in A$, $\lambda \in k$.

Definition:

A Lie algebra is an algebra L with a multiplication $[?,?]: L \times L \longrightarrow L$ satisfying:

(1) Skew-symmetry: $[x,x] = 0$ for all $x \in L$,

(2) Jacobi identity: $[[x,y],z] + [[y,z],x] + [[z,x],y] = 0$ for all $x, y, z \in L$.

Example:

(i) The general linear Lie algebra $gl_n(k)$ are all $n \times n$ matrices over k with bracket:

$$[M,N] := MN - NM \quad \text{for all } M, N \in gl_n(k).$$

(ii) The special linear Lie algebra $sl_n(k)$ are all $n \times n$ matrices over k with zero trace:

$$sl_n(k) := \{ M \in gl_n(k) : tr(M) = 0 \} \subseteq gl_n(k).$$ It is a Lie subalgebra of $gl_n(k)$ with the bracket that it inherits.

(iii) Let V be a vector space of dimension n over a field k, let $L = End(V)$ be the linear endomorphisms of V. Then L is an associative algebra with multiplication the composition of functions. Taking L with bracket:

$$[f,g] := fg - gf \quad \text{for all } f, g \in L$$ gives the Lie algebra $gl(V)$.

Definition:

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Let L, M be Lie algebras, we say that a linear map $\varphi: L \longrightarrow M$ is a Lie homomorphism if:

$\varphi([x,y]) = [\varphi(x), \varphi(y)]$ for all $x, y \in L$.

Example:

Let $V = k^n$, fix $\mathcal{B}$ a basis of V, then:

$$\begin{aligned} \phi: \mathfrak{gl}(V) &\longrightarrow \mathfrak{gl}_n(k) \\ T &\longmapsto [T]_{\mathcal{B}} \end{aligned}$$

is a Lie isomorphism.

Definition:

A finite dimensional representation of a Lie algebra L is a Lie homomorphism:

$\rho: L \to \mathfrak{gl}(V)$ where V is a finite dimensional vector space.

A vector space V is a module over L a Lie algebra if there exists a map $\gamma: L \times V \longrightarrow V$ satisfying:

(1) $\gamma(\ell_1 + \ell_2, v) = \gamma(\ell_1, v) + \gamma(\ell_2, v)$,

(2) $\gamma(\ell, v_1 + v_2) = \gamma(\ell, v_1) + \gamma(\ell, v_2)$,

(3) $\lambda\gamma(\ell, v) = \gamma(\lambda\ell, v) = \gamma(\ell, \lambda v)$,

(4) $\gamma([\ell_1, \ell_2], v) = \gamma(\ell_1, \gamma(\ell_2, v)) - \gamma(\ell_2, \gamma(\ell_1, v))$ for all $\ell, \ell_1, \ell_2 \in L$, $v, v_1, v_2 \in V$, $\lambda \in k$.

Remark:

Given a representation $\rho: L \to \mathfrak{gl}(V)$, then $\begin{aligned} \gamma: L \times V &\longrightarrow V \\ (\ell, v) &\longmapsto \rho(\ell)(v) \end{aligned}$ makes V a module over L.

Given V a module over L, then $\begin{aligned} \rho: L &\longrightarrow \mathfrak{gl}(V) \\ \ell &\longmapsto \left(\begin{aligned} \rho(\ell): V &\longrightarrow V \\ v &\mapsto \gamma(\ell, v) \end{aligned}\right) \end{aligned}$ makes V a representation of L.

Representations of $\mathfrak{sl}_2(\mathbb{C})$.

Let $L = \mathfrak{sl}_2(\mathbb{C})$ with basis $\mathcal{B}$: $e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$,

and brackets: $[e, f] = h$, $[e, h] = -2e$, $[f, h] = 2f$.

Theorem: There is exactly one irreducible module of L (up to isomorphism) for each dimension.

Definition:

For each $d \geq 0$, let: $V_d := \mathrm{Span}(x^d, x^{d-1}y, \ldots, xy^{d-1}, y^d) \subseteq \mathbb{C}[x,y]$, $\dim(V_d) = d+1$,

the subspace of all homogeneous polynomials of degree d.

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Let: $\varphi : L \longrightarrow \mathfrak{gl}(V)$, so explicitly: $\varphi(e) : V \longrightarrow V$, and extend by linearity.

$$e \longmapsto x\frac{\partial}{\partial y} \qquad\qquad x^a y^b \longmapsto b x^{a+1} y^{b-1} \quad (b \geq 1)$$

$$x^d \longmapsto 0$$

$$f \longmapsto y\frac{\partial}{\partial x} \qquad\qquad \varphi(f) : V \longrightarrow V,$$

$$x^a y^b \longmapsto a x^{a-1} y^{b+1} \quad (a \geq 1)$$

$$h \longmapsto x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} \qquad\qquad y^d \longmapsto 0$$

$$\varphi(h) : V \longrightarrow V.$$

$$x^a y^b \longmapsto (a-b) x^a y^b$$

Proposition: The map $\varphi : L \longrightarrow \mathfrak{gl}(V)$ is a representation of L. Hence V_d is a $(d+1)$-dimensional module of L.

Proof: It suffices to check:

(1) $[\varphi(h), \varphi(e)] = \varphi([h,e]) = 2\varphi(e)$,

(2) $[\varphi(h), \varphi(f)] = \varphi([h,f]) = -2\varphi(f)$,

(3) $[\varphi(e), \varphi(f)] = \varphi([e,f]) = \varphi(h)$. $\square$.

Matrix representations:

Let $\mathcal{B} = \{x^d, x^{d-1}y, \dots, xy^{d-1}, y^d\}$ be a basis for V_d. Then:

$$L \xrightarrow{\varphi} \mathfrak{gl}(V) \xrightarrow{\phi} \mathfrak{gl}_n(k)$$

$$e \longmapsto \varphi(e) \longmapsto \begin{bmatrix} 0 & 1 & & & \\ & 0 & 2 & & \\ & & \ddots & \ddots & \\ & & & 0 & d \\ & & & & 0 \end{bmatrix} = [\varphi(e)]_{\mathcal{B}}$$

$$f \longmapsto \varphi(f) \longmapsto \begin{bmatrix} 0 & & & & \\ d & 0 & & & \\ & d-1 & \ddots & & \\ & & \ddots & 0 & \\ & & & 1 & 0 \end{bmatrix} = [\varphi(f)]_{\mathcal{B}}$$

$$h \longmapsto \varphi(h) \longmapsto \begin{bmatrix} d & & & & \\ & d-2 & & & \\ & & \ddots & & \\ & & & -(d-2) & \\ & & & & -d \end{bmatrix} = [\varphi(h)]_{\mathcal{B}}$$

Example:

$d=0$: Trivial representation: $e, f, h \mapsto 0$.

$d=1$: $e \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $f \mapsto \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $h \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

$\underline{d=1}$: $\quad e \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad f \mapsto \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad h \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$

$\underline{d=2}$: $\quad e \mapsto \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \quad f \mapsto \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \quad h \mapsto \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$; isomorphic to the adjoint representation.

<u>Proposition:</u> The module V_d is irreducible.

<u>Proof:</u> Suppose $U \subseteq V_d$ is a non-zero submodule. Then $\varphi(e)(U) \subseteq U$, $\varphi(f)(U) \subseteq U$, $\varphi(h)(U) \subseteq U$.

Since $[\varphi(h)]_{\mathcal{B}} = \begin{bmatrix} d & & & & \\ & d-2 & & & \\ & & \ddots & & \\ & & & -(d-2) & \\ & & & & -d \end{bmatrix}$ has $d+1$ distinct eigenvalues, the eigenvalues of $\varphi(h)|_U$

are also distinct, and U contains an eigenvector for $\varphi(h)$. Hence $x^a y^b \in U$ for some basis element in $\mathcal{B}$. Applying $\varphi(e)$ successively we get: $x^{a+1}y^{b-1}, x^{a+2}y^{b-2}, \ldots, x^d \in U$. Applying $\varphi(f)$ successively we get: $x^{a-1}y^{b+1}, x^{a-2}y^{b+2}, \ldots, y^d \in U$. Hence $U = V_d$. $\square$.

<u>Lemma:</u> Let V be a finite dimensional module over L.

(1) If $v \in V$ with $hv = \lambda v$ then: $h(ev) = (\lambda+2)ev$, $h(fv) = (\lambda-2)fv$.

(2) There exists $w \in V$ a non-zero eigenvector of h such that $ew = 0$.

<u>Proof:</u> (1) We have:

$$h(ev) = e(hv) + [h,e]v = e(\lambda v) + 2ev = (\lambda+2)ev,$$
$$h(fv) = f(hv) + [h,f]v = f(\lambda v) - 2fv = (\lambda-2)fv.$$

(2) Since $\mathbb{C}$ is the base field, the linear map $u \mapsto hu$ has an eigenvector, say $hv = \lambda v$. Consider: $v, ev, e^2v, \ldots$, if all of these are non-zero, by (1) we have eigenvectors of h with distinct eigenvalues, so they are linearly independent. Since V is finite dimensional, there is an $n \in \mathbb{N}$ with $e^n v \neq 0$ but $e^{n+1} v = 0$. Set $w = e^n v$. $\square$

<u>Theorem:</u> Let V be an irreducible finite dimensional module over L. Then $V \cong V_d$ for some $d \geq 0$.

<u>Proof:</u> By (2) above, there exists a non-zero $w \in V$ with $hw = \lambda w$, $ew = 0$. By the proof of (2) above, there exists $d \geq 0$ such that $f^d w \neq 0$, $f^{d+1} w = 0$.

<u>Step 1:</u> The elements $w, fw, \ldots, f^d w$ form a basis of V consisting of eigenvectors of h with eigenvalues $\lambda, \lambda-2, \ldots, \lambda-2d$

Step 1: The elements $w, fw, \dots, f^d w$ form a basis of V consisting of eigenvectors of h with eigenvalues $\lambda, \lambda-2, \dots, \lambda-2d$.

Linear independence and being eigenvectors of h follows from the Lemma above. To show they span V, set $\mathcal{U} := \mathrm{Span}(w, fw, \dots, f^d w)$. This is a submodule: $f(\mathcal{U}) \subseteq \mathcal{U}$, $h(\mathcal{U}) \subseteq \mathcal{U}$, and $ef^n w \in \mathrm{Span}(w, fw, \dots, f^{n-1} w)$ for all $n \leq d$ by induction:

$n=0$: $ew = 0 \in \mathrm{Span}(\emptyset)$

$n-1$: $ef^{n-1} w \in \mathrm{Span}(w, fw, \dots, f^{n-2} w)$

n: $ef^n w = e(f(f^{n-1} w)) = (fe + [e,f])(f^{n-1} w) = (fe+h) f^{n-1} w = fef^{n-1} w + hf^{n-1} w$

where $ef^{n-1} w \in \mathrm{Span}(w, fw, \dots, f^{n-2} w)$ implies $fef^{n-1} w \in \mathrm{Span}(w, fw, \dots, f^{n-1} w)$ and $f^{n-1} w \in \mathrm{Span}(w, fw, \dots, f^{n-1} w)$ implies $hf^{n-1} w \in \mathrm{Span}(w, fw, \dots, f^{n-1} w)$.

Hence $e(\mathcal{U}) \subseteq \mathcal{U}$. Since V is irreducible, $\mathcal{U} = V$.

Step 2: Consider the basis $\mathcal{B} = \{w, fw, \dots, f^d w\}$ of V. Then: $[h]_{\mathcal{B}} = \begin{bmatrix} \lambda & & \\ & \lambda-2 & \\ & & \ddots \\ & & & \lambda-2d \end{bmatrix}$, and $h = [e,f] \in \mathrm{Span}\{[x,y] : x,y \in L\} = [L,L] = L$ so $\mathrm{tr}([h]_{\mathcal{B}}) = 0$.

Hence: $\lambda + \lambda - 2 + \dots + \lambda - 2d = 0$ so $(d+1)\lambda = d(d+1)$ so $\lambda = d$.

Step 3: We have V with basis $\{w, fw, \dots, f^d w\}$ and V_d with basis $\{x^d, fx^d, \dots, f^d x^d\}$. Both consist of eigenvectors of h with eigenvalues $d, d-2, \dots, -d$. Let: $\varphi : V \longrightarrow V_d$, $f^n w \longmapsto f^n x^d$.

This is an isomorphism of modules:

$$f\varphi(f^n w) = f(f^n x^d) = f^{n+1} x^d = \varphi(f^{n+1} w),$$

$$h\varphi(f^n w) = h(f^n x^d) = (d-2n) f^n x^d = (d-2n)\varphi(f^n w) = \varphi((d-2n) f^n w) = \varphi(hf^n w),$$

and $e\varphi(f^n w) = \varphi(ef^n w)$ by induction:

$n=0$: $e\varphi(w) = ex^d = 0 = \varphi(0) = \varphi(ew)$.

$n-1$: $e\varphi(f^{n-1} w) = \varphi(ef^{n-1} w)$.

n: $\varphi(ef^n w) = \varphi((ef) f^{n-1} w) = \varphi((fe + [e,f]) f^{n-1} w) = \varphi((fe+h) f^{n-1} w) =$

$= f\varphi(ef^{n-1} w) + h\varphi(f^{n-1} w) = fe\varphi(f^{n-1} w) + h\varphi(f^{n-1} w) =$

$= (fe+h)\varphi(f^{n-1} w) = (fe + [e,f])\varphi(f^{n-1} w) = (ef)\varphi(f^{n-1} w) = e\varphi(f^n w)$. $\square$.