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Lectures on Quantum Groups by J. C. Jantzen.

Introduction:

There is still no rigorous, universally accepted definition of the term quantum group.
It does include skew deformations of Hopf algebras.

Example:

$k[x,y]$ is $k\langle x,y\rangle / (xy-yx)$ the associative algebra with generators x, y and relation $yx = xy$.

For each $q \in k$ let $k_q[x,y]$ be the associative algebra with generators x, y and relation $yx = qxy$: $k\langle x,y\rangle / (yx - qxy)$. All monomials $x^m y^n$ are a basis of $k_q[x,y]$ over k and: $x^m y^n x^r y^s = q^{nr} x^{m+r} y^{n+s}$. This is called the quantum plane.

So $k_q[x,y]$ are a family of algebras where the multiplication depends "nicely" on the parameter q, and $k_1[x,y] = k[x,y]$ recovers the original algebra. This is called a deformation of $k[x,y]$.

In the study of quantum groups one deals with deformations $U_q(\mathfrak{g})$, the so called quantized enveloping algebra, related to the enveloping algebra $U(\mathfrak{g})$ of a semisimple Lie algebra $\mathfrak{g}$ (keep $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ in mind as the motivating example). These are also called small quantum groups.

Applications:

To construct solutions to the quantum Yang-Baxter equation, in theoretical physics, low-dimensional topology and knot theory, representation theory of algebraic groups, ...

Gaussian Binomial Coefficients.

Definition: Let v be an indeterminate, consider $\mathbb{Z}[v, v^{-1}] \subseteq \mathbb{Q}(v)$. Set:

$$[a] := \frac{v^a - v^{-a}}{v - v^{-1}} \quad \text{for all } a \in \mathbb{Z}.$$

Rmk:

(1) $[0] = 0$.

(2) $[a] \neq 0$ for $a \neq 0$.

(3) $[-a] = -[a]$.

(4) $[a] = v^{a-1} + v^{a-3} + \cdots + v^{-a+3} + v^{-a+1}$ for $a > 0$.

(5) $[a] \in \mathbb{Z}[v, v^{-1}]$.

Definition: The Gaussian binomial coefficients are:

$$\begin{bmatrix} a \\ n \end{bmatrix} := \frac{[a][a-1]\cdots[a-n+1]}{[1][2]\cdots[n]} \quad \text{for all } a, n \in \mathbb{Z} \text{ with } n > 0; \quad \begin{bmatrix} a \\ 0 \end{bmatrix} := 1.$$

Rmk:

(1) $\begin{bmatrix} a \\ 1 \end{bmatrix} = [a]$.

(2) $\begin{bmatrix} n \\ n \end{bmatrix} = 1$.

(3) $\begin{bmatrix} a \\ n \end{bmatrix} = 0$ for $0 \leq a < n$.

(4) $\begin{bmatrix} a \\ n \end{bmatrix} = (-1)^n \begin{bmatrix} -a+n-1 \\ n \end{bmatrix}$ for all $a, n \in \mathbb{Z}$. In particular $\begin{bmatrix} -1 \\ n \end{bmatrix} = (-1)^n$ for all $n \in \mathbb{Z}$.

Definition: Set: $[0]! := 1$, $[n]! := [1][2]\cdots[n]$ for $n > 0$.

Rmk:

(1) $\begin{bmatrix} a \\ n \end{bmatrix} = \dfrac{[a]!}{[n]![a-n]!}$ for all $a, n \geq 0$.

(2) $\begin{bmatrix} a+1 \\ n \end{bmatrix} = v^{-n} \begin{bmatrix} a \\ n \end{bmatrix} + v^{a-n+1} \begin{bmatrix} a \\ n-1 \end{bmatrix}$, so $\begin{bmatrix} a \\ n \end{bmatrix} \in \mathbb{Z}[v, v^{-1}]$ for all $a, n \in \mathbb{Z}$ with $n \geq 0$.

That is, all Gaussian binomial coefficients are in $\mathbb{Z}[v, v^{-1}]$.

(3) $\begin{bmatrix} a+1 \\ n \end{bmatrix} = v^{n} \begin{bmatrix} a \\ n \end{bmatrix} + v^{-a+n-1} \begin{bmatrix} a \\ n-1 \end{bmatrix}$.

(4) $\displaystyle\sum_{i=0}^{r} (-1)^i v^{i(r-1)} \begin{bmatrix} r \\ i \end{bmatrix} = 0$.

(5) $\displaystyle\sum_{i=0}^{r} (-1)^i v^{-i(r-1)} \begin{bmatrix} r \\ i \end{bmatrix} = 0$.

(6) If k is a ring with 1 and $q \in k$ is invertible, then there is a unique ring homomorphism:

$\mathbb{Z}[v, v^{-1}] \longrightarrow k$

(6) If R is a ring with 1 and $q \in R$ is invertible, then there is a unique ring homomorphism:

$$\begin{aligned} \mathbb{Z}[v,v^{-1}] &\longrightarrow k \\ v &\longmapsto q \\ v^{-1} &\longmapsto q^{-1} \end{aligned}$$

that allows us to see $[a], [n]!, \begin{bmatrix} a \\ n \end{bmatrix} \in k$.
$\quad\; \| \quad\; \| \quad\; \|$
$[a]_q \quad [n]_q^! \quad \begin{bmatrix} a \\ n \end{bmatrix}_q$

(7) $[a]_1 = a$, $[n]_1^! = n!$, $\begin{bmatrix} a \\ n \end{bmatrix}_1 = \binom{a}{n}$.

The Quantized Enveloping Algebra $U_q(sl_2)$.

Definition: Let k be a field, fix $q \in k$ with $q \neq 0$, $q^2 \neq 1$. Then $U_q(sl_2)$ is the associative unital algebra over k generated by E, F, K, K^{-1} with relations:

(R1) $KK^{-1} = 1 = K^{-1}K$.

(R2) $KEK^{-1} = q^2 E$.

(R3) $KFK^{-1} = q^{-2} F$.

(R4) $EF - FE = \dfrac{K - K^{-1}}{q - q^{-1}}$.

We may abuse $U := U_q(sl_2)$.

Goal: $U_q(sl_2)$ is supposed to be a quantum analogue of $U(sl_2)$. We want to show:

1. $U_q(sl_2)$ has a PBW type basis.
2. $U_q(sl_2)$ has no zero divisors.

Definition: The algebra $sl_2(\mathbb{C})$ is spanned by $e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and satisfy:

$[h,e] = 2e$, $[h,f] = -2f$, $[e,f] = h$.

The universal enveloping algebra of $sl_2(\mathbb{C})$ is generated by E, F, H with relations:

$HE - EH = 2E$, $HF - FH = -2F$, $EF - FE = H$.

Theorem (Poincaré-Birkhoff-Witt): Let $\{v_i\}_{i=1}^n$ be a basis of a finite dimensional Lie algebra L, then the set of all monomials: $v_{i_1}^{j_1} \cdots v_{i_m}^{j_m}$ with $i_1 < \cdots < i_m$ and $j_1, \dots, j_m \geq 0$ is a basis of $U(L)$.

Lemma: a) There is a unique automorphism: $\omega: U \longrightarrow U$, it satisfies $\omega^2 = 1$.

$$\begin{aligned} E &\longmapsto F \\ F &\longmapsto E \\ K &\longmapsto K^{-1} \end{aligned}$$

b) There is a unique automorphism: $\tau: U^{op} \longrightarrow U^{op}$, it satisfies $\tau^2 = 1$.

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$$E \longmapsto E$$
$$F \longmapsto F$$
$$K \longmapsto K^{-1}$$

__Proof:__ a) We have to check that the reordering (F, E, K^{-1}, K) satisfy (R1), (R2), (R3), (R4).

The uniqueness follows from E, F, K, K^{-1} generating U. Clearly $\omega^2 = 1$.

b) __Definition:__ A^{op} is the opposite algebra of A: it has its same underlying vector space and multiplication $a \cdot_{op} b := ba$.

We have to check that the reordering (E, F, K^{-1}, K) satisfy (R1), (R2), (R3), (R4) in U^{op}.

(R4) $E \cdot_{op} F - F \cdot_{op} E = FE - EF = \dfrac{K^{-1} - K}{q - q^{-1}}$. The rest are analogous.

The uniqueness follows from E, F, K, K^{-1} generating U. Clearly $\tau^2 = 1$. $\square$.

__Definition:__ Let: $[K; a] := \dfrac{Kq^a - K^{-1}q^{-a}}{q - q^{-1}}$ for all $a \in \mathbb{Z}$.

__Rmk:__

(1) We can write (R4) as: $EF - FE = [K; 0]$.

(2) $[b+c][K; a] = [b][K; a+c] + [c][K; a-b]$ for all $a, b, c \in \mathbb{Z}$, where $v = q$.

(3) $[K; a]E = E[K; a+2]$

(4) $[K; a]F = F[K; a-2]$

(5) The automorphism ω satisfies: $\omega([K; a]) = -[K; -a]$ for all $a \in \mathbb{Z}$.

(6) $EF^s = F^sE + [s]F^{s-1}[K; 1-s]$ for all $s \in \mathbb{Z}$, $s > 0$.

(7) $FE^r = E^rF - [r]E^{r-1}[K; r-1]$ for all $r \in \mathbb{Z}$, $r > 0$.

__Lemma:__ The algebra U is spanned as a vector space over k by all monomials:

$F^s K^n E^r$ with $r, s, n \in \mathbb{Z}$, $r, s \geq 0$.

__Proof:__ The span of these monomials is stable under multiplication by all generators of U:

$F F^s K^n E^r = F^{s+1} K^n E^r$,

$K F^s K^n E^r = q^{-2s} F^s K^{n+1} E^r$,

$K^{-1} F^s K^n E^r = q^{2s} F^s K^{n-1} E^r$

$KF^sK^nE^r = q^{-2s}F^sK^{n+1}E^r$,

$K^{-1}F^sK^nE^r = q^{2s}F^sK^{n-1}E^r$,

$EF^sK^nE^r = F^sEK^nE^r + [s]F^{s-1}[K;1-s]K^nE^r = q^{-2n}F^sK^nE^{r+1} + [s]F^{s-1}[K;1-s]K^nE^r$.

When $s=0$, the last term is zero.

When $s>0$, we can write $[K;1-s]K^n$ as a polynomial in K and K^{-1}.

Then the span of these monomials is stable under multiplication with any element in U, so it contains $U \cdot F^0K^0E^0 = U \cdot 1 = U$. $\square$.

Theorem: The monomials $F^sK^nE^r$ with $r,s,n \in \mathbb{Z}$, $r,s \geq 0$ are a basis of U.

Proof: Since these monomials span U, it only remains to show that they are linear independent. Consider $k[x,y,z]$ and its localization $A = k[x,y,z,z^{-1}]$. Here all monomials $y^sz^nx^r$ with $r,s,n \in \mathbb{Z}$, $r,s \geq 0$ are a basis of A by commutativity. Define endomorphisms of A by:

$$f : A \longrightarrow A, \quad y^sz^nx^r \longmapsto y^{s+1}z^nx^r$$

$$e : A \longrightarrow A, \quad y^sz^nx^r \longmapsto q^{-2n}y^sz^nx^{r+1} + [s]y^{s-1}\frac{zq^{1-s} - z^{-1}q^{s-1}}{q-q^{-1}}z^nx^r$$

$$h : A \longrightarrow A, \quad y^sz^nx^r \longmapsto q^{-2s}y^sz^{n+1}x^r$$

being invertible with inverse

$$h^{-1} : A \longrightarrow A, \quad y^sz^nx^r \longmapsto q^{2s}y^sz^{n-1}x^r.$$

We can now check that (e,f,h,h^{-1}) satisfy (R1), (R2), (R3), (R4). Therefore there is a homomorphism:

$$\begin{aligned} U &\longrightarrow \mathrm{End}_k(A). \\ E &\longmapsto e \\ F &\longmapsto f \\ K^{\pm} &\longmapsto h^{\pm} \\ F^sK^nE^r &\longmapsto f^sh^ne^r \end{aligned}$$

Since: $f^sh^ne^r(1) = f^sh^n(x^r) = f^s(z^nx^r) = y^sz^nx^r$ for all $r,s,n \in \mathbb{Z}$, $r,s \geq 0$, we have that the $f^sh^ne^r$ are linearly independent. Thus $F^sK^nE^r$ are linearly independent. $\square$

Definition: Let U^+ be the subalgebra of U generated by E.

Definition: Let U^+ be the subalgebra of U generated by E.
Let U_- be the subalgebra of U generated by F.
Let U^0 be the subalgebra of U generated by K, K^{-1}.

Rmk: The E^r with $r \in \mathbb{Z}$, $r \geq 0$, are a basis of U^+, since by the Theorem above these elements are linearly independent. Similarly F^s with $s \in \mathbb{Z}$, $s \geq 0$, are a basis of U_-. Similarly K^n with $n \in \mathbb{Z}$ are a basis of U^0. Hence:

$U^+ \cong k[x]$, $U_- \cong k[y]$ (they are isomorphic to a polynomial algebra in one variable),

$U^0 \cong k[z, z^{-1}]$ (it is isomorphic to the localization of a polynomial algebra in one variable).

Definition: Let: $\gamma_i : U^0 \longrightarrow U^0$ for $i \in \mathbb{Z}$.
$$K \longmapsto q^i K$$

Rmk:

(1) $\gamma_0 = 1_{U^0}$.

(2) $\gamma_i \gamma_j = \gamma_{i+j}$ for all $i, j \in \mathbb{Z}$.

(3) $\gamma_i([K; a]) = [K; a+i]$ for all $i, a \in \mathbb{Z}$.

(4) For all $u \in U^0$: $\quad uE = E\gamma_2(u)$ and $uF = F\gamma_{-2}(u)$.

Lemma: For all $r, s \in \mathbb{Z}$, $r, s \geq 0$, we have:
$$E^r F^s = \sum_{i=0}^{\min(r,s)} \begin{bmatrix} r \\ i \end{bmatrix} \begin{bmatrix} s \\ i \end{bmatrix} [i]! \, F^{s-i} \left(\prod_{j=1}^{i} [K; i-(r+s)+j] \right) E^{r-i}.$$

Proof: By induction. $\square$.

It is reasonable to prove that: $E^r F^s = \sum_{i=0}^{\min(r,s)} F^{s-i} u_i E^{r-i}$ for some $u_i \in U^0$.

Proving that $u_i = \begin{bmatrix} r \\ i \end{bmatrix} \begin{bmatrix} s \\ i \end{bmatrix} [i]! \left(\prod_{j=1}^{i} [K; i-(r+s)+j] \right)$ is less reasonable.

Proposition: The algebra U has no zero divisors.

Proof: Any non-zero $u \in U$ can be written as a sum of a term:

$F^s h E^r$ with $h \in U^0$, $h \neq 0$, $r, s \in \mathbb{Z}$, $r, s \geq 0$

$F^s h E^r$ with $h \in U^0$, $h \neq 0$, $r,s \in \mathbb{Z}$, $r,s \geq 0$

and of terms:

$F^{s'} h' E^{r'}$ with $h' \in U^0$, $r', s' \in \mathbb{Z}$, $r', s' \geq 0$

where either $s' < s$ or $s' = s$ and $r' < r$ (we call $F^s h E^r$ the leading term of u).

We have by the Lemma above:

$$(F^s h E^r)(F^p h' E^m) = \sum_{i=0}^{\min(r,p)} F^s h F^{p-i} h_i E^{r-i} h' E^m \quad \text{with suitable } h_i \in U^0,\ h_0 = 1.$$

Using γ_i we have:

$$(F^s h E^r)(F^p h' E^m) = \sum_{i=0}^{\min(r,p)} F^{s+p-i} \gamma_{2(i-p)}(h) h_i \gamma_{2(i-r)}(h') E^{r-i+m}.$$

So if $h \neq 0$ and $h' \neq 0$, since U^0 is integral domain and γ_j are automorphisms, we have $\gamma_{-2p}(h)\gamma_{-2r}(h') \neq 0$ and thus the leading term of $(F^s h E^r)(F^p h' E^m)$ is

$$F^{s+p} \gamma_{-2p}(h) \gamma_{-2r}(h') E^{r+m}.$$

Hence if $u, v \in U$, $u \neq 0 \neq v$ have leading terms $F^s h E^r$ and $F^p h' E^m$ respectively, then uv has leading term

$$F^{s+p} \gamma_{-2p}(h) \gamma_{-2r}(h') E^{r+m}, \quad \text{so in particular } uv \neq 0.$$

$\square$.

<u>Rmk:</u>

The product of $F^s k^n E^r$ and $F^p k^\ell E^m$ is, by the proof of the above Theorem, a linear combination of monomials $F^i k^h E^j$ with $j - i = (r-s) + (m-p)$. Hence we can see U as a graded algebra where each $F^s K^n E^r$ is homogeneous of degree $r-s$:

$$U^{r-s} \times U^{m-p} \longrightarrow U^{r-s+m-p}$$
$$(F^s k^n E^r, F^p k^\ell E^m) \longmapsto F^i k^h E^j \quad \text{with } j-i = r-s+m-p.$$

We can also see this grading directly from our construction. A free algebra can be graded by assigning degrees to the generators. In our case setting:

$\deg(E) = 1$, $\deg(F) = -1$, $\deg(K) = 0 = \deg(K^{-1})$

we see that the relations (R1) (R2) are homogeneous of degree 0, and they generate a graded

we see that the relations (R1) are homogeneous of degree 0, and they generate a graded
(R2) 1
(R3) −1
(R4) 0

ideal in the free algebra. The factor algebra U inherits a grading, and clearly each $F^s K^n E^r$ is homogeneous of degree $r-s$ for this grading.

If $u \in U$ is homogeneous of degree i, then $KuK^{-1} = q^{2i} u$.

If q is <u>not</u> a root of unity, then q^i, $i \in \mathbb{Z}$, are all distinct, and the graded pieces of U are exactly the eigenspaces of the map:
$$\begin{aligned} U &\longrightarrow U \\ u &\longmapsto KuK^{-1} \end{aligned}.$$

If q is a root of unity, then the grading is finer than the eigenspace decomposition.