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Representations of $U_q(sl_2)$.

Goal: Look at finite dimensional representations and determine the center of U

a) If q not root of unity, U behaves like $U(sl_2)$ over a field of characteristic 0
b) If q root of unity, U behaves like $U(sl_2)$ over a field of prime characteristic
c) For $k=\mathbb{C}$ and q a primitive p-th root of unity for $p\geq 3$ prime, we get a representation theory that looks like the representation theory of sl_2 over an algebraically closed field of characteristic p.

Definition: Let k be a field, fix $q\in k$ with $q\neq 0$, $q^2\neq 1$. Then $U_q(sl_2)$ is the associative unital algebra over k generated by E, F, K, K^{-1} with relations

(R1) $KK^{-1}=1=K^{-1}K$.

(R2) $KEK^{-1}=q^2E$

(R3) $KFK^{-1}=q^{-2}F$.

(R4) $EF-FE=\dfrac{K-K^{-1}}{q-q^{-1}}$

We may abuse $U:=U_q(sl_2)$.

Proposition: Suppose that q is not a root of unity, let M be a finite dimensional U-module. Then there are integers $r,s>0$ with $E^rM=0$ and $F^sM=0$.

Definition Let M be a U-module, set for all $\lambda\in k$, $\lambda\neq 0$:

$$M_\lambda=\{m\in M \mid Km=\lambda m\}.$$

We call M_λ a weight space of M, we call the λ with $M_\lambda\neq 0$ weights of M

Proposition: Suppose that q is not a root of unity and that $\mathrm{char}(k)\neq 2$, let M be a finite dimensional U-module. Then M is the direct sum of its weight spaces, and all weights of M have the form $\pm q^a$ with $a\in\mathbb{Z}$.

Definition For each $\lambda\in k$, $\lambda\neq 0$, set. $M(\lambda):=\dfrac{U}{(UE+U(K-\lambda))}$.

Remark This is an infinite dimensional U-module with basis $m_0, m_1, m_2, \dots$ satisfying

$$Km_i=\lambda q^{-2i}m_i,$$
$$Fm_i=m_{i+1},$$
$$Em_i=\begin{cases}0 & \text{if } i=0,\\ [i]\cdot\dfrac{\lambda q^{1-i}-\lambda^{-1}q^{i-1}}{q-q^{-1}}\cdot m_{i-1} & \text{if } i\neq 0.\end{cases}$$

It also has the following universal property:
If M is a U-module and $m\in M$ a vector with $Em=0$ and $Km=\lambda m$, then there is a unique homomorphism of U-modules $\varphi: M(\lambda)\longrightarrow M$ with $\varphi(m_0)=m$.

Proposition: Suppose that q is not a root of unity, let $\lambda\in k$, $\lambda\neq 0$.

Infinite dimensional irreducible representations.

a) If $\lambda\neq\pm q^n$ for all $n\in\mathbb{N}$, then $M(\lambda)$ is simple.
b) If $\lambda=\pm q^n$ for some $n\in\mathbb{N}$, then the m_i with $i\geq n+1$ span a submodule of $M(\lambda)$ isomorphic to $M(q^{-2(n+1)}\lambda)$ This is the only submodule of $M(\lambda)$ different from 0 and $M(\lambda)$

Theorem: Suppose that q is not a root of unity. There are for each $n\in\mathbb{N}$ a simple U-module $L(n,+)$ with basis $m_0, m_1, \dots, m_n$ such that for all $0\leq i\leq n$:
$L(n,-)$ $\quad m_0', m_1', \dots, m_n'$

Infinitely many finite

$$K m_i = q^{n-2i} m_i, \qquad K m_i' = -q^{n-2i} m_i',$$

dimensional irreducible representations.

$$F m_i = \begin{cases} m_{i+1} & \text{if } i<n, \\ 0 & \text{if } i=n, \end{cases} \qquad F m_i' = \begin{cases} m_{i+1}' & \text{if } i<n, \\ 0 & \text{if } i=n, \end{cases}$$

$$E m_i = \begin{cases} [i][n+1-i] m_{i-1} & \text{if } i>0, \\ 0 & \text{if } i=0. \end{cases} \qquad E m_i' = \begin{cases} -[i][n+1-i] m_{i-1}' & \text{if } i>0, \\ 0 & \text{if } i=0. \end{cases}$$

Moreover each simple U-module of dimension $n+1$ is isomorphic to $L(n,+)$ or $L(n,-)$.

Definition. Set : $C = FE + \dfrac{Kq + K^{-1}q^{-1}}{(q-q^{-1})^2}$. This is called a Casimir element of U

Philosophy: The Casimir element will commute with all elements in U, and it will act on U-modules by the usual module action. By Schur's Lemma, in any irreducible representation of U this Casimir element will act proportionally to the identity. This constant of proportionality can be used to clarify representations of U (we may need $k=\bar{k}$)

Theorem: (Schur's Lemma) Given M, N two simple modules over a ring R, then any homomorphism $f: M \longrightarrow N$ of R-modules is either invertible or zero

Lemma:

a) The element C is central in U.

b) C acts on each $M(\lambda)$ as a scalar multiplication by $\dfrac{\lambda q + \lambda^{-1} q^{-1}}{(q-q^{-1})^2}$.

c) C acts on $M(\lambda)$ and $M(\mu)$ by the same scalar if and only if $\lambda = \mu$ or $\lambda = \mu^{-1} q^{-2}$

Lemma: Suppose that q is not a root of unity, let M be a finite dimensional U-module that is the direct sum of its weight spaces (in particular this always happens in $\mathrm{char}(k) \neq 2$). Then M is semisimple.

Complete reducibility of (almost all) finite dimensional representations.

Remark. Suppose that q is a root of unity: $q^\ell = 1$, then $[\ell] = 0$ hence $[i]^! = 0$ whenever $i \geq \ell > 2$.

Proposition. If q is a primitive ℓ-th root of unity ($\ell \in \mathbb{Z}$, $\ell \geq 3$) then $E^\ell, F^\ell, K^\ell, K^{-\ell}$ are in the center of U.

Proof: We only need to show that they commute with the generators of U.

(i) $K^\ell, K^{-\ell}$ central: $\quad K^\ell E K^{-\ell} = q^{2\ell} E = E$ and $K^\ell F K^{-\ell} = q^{-2\ell} F = F$.

(ii) E^ℓ central $\quad K E^\ell K^{-1} = q^{2\ell} E^\ell = E^\ell$ and $F E^\ell = E^\ell F - [\ell] E^{\ell-1} [K; \ell-1] = E^\ell F$

(iii) F^ℓ central: $\quad K F^\ell K^{-1} = q^{-2\ell} F^\ell = F^\ell$ and $E F^\ell = F^\ell E + [\ell] F^{\ell-1} [K; 1-\ell] = F^\ell E$. □

Rmk. If $\ell = 2\ell'$ then $0 = [\ell] = [2\ell'] = \dfrac{q^{2\ell'} - q^{-2\ell'}}{q-q^{-1}} = q^{\ell'} \cdot \dfrac{q^{\ell'} - q^{-\ell'}}{q-q^{-1}}$ so $[\ell'] = 0$ and $E^{\ell'}, F^{\ell'}, K^{\ell'}, K^{-\ell'}$ are also central. We can restrict ourselves to considering q a primitive ℓ-th root of unity with ℓ-odd. The even case will follow by replacing ℓ with a fraction $\dfrac{\ell}{2^i}$ for some $i \in \mathbb{N}$.

Definition. Suppose that q is a primitive ℓ-root of unity with ℓ-odd, $\ell \geq 3$. Set for $b, \lambda \in k$, $\lambda \neq 0$:

$$Z_b(\lambda) := \frac{M(\lambda)}{U(m_\ell - b m_0)}.$$

Rmk:

We have:

$$E m_\ell = [\ell] \cdot \frac{\lambda q^{1-i} - \lambda^{-1} q^{i-1}}{q-q^{-1}} \cdot m_{\ell-1} = 0, \qquad K m_\ell = \lambda q^{-2\ell} m_\ell = \lambda m_\ell$$

Hence:

$$E(m_\ell - b m_0) = 0, \qquad K(m_\ell - b m_0) = \lambda (m_\ell - b m_0)$$

and thus $U(m_\ell - b m_0)$ is spanned by all $F^i(m_\ell - b m_0) = m_{\ell+i} - b m_i$ with $i \geq 0$. In particular the images of the m_j with $j<\ell$ in $Z_b(\lambda)$ are a basis of $Z_b(\lambda)$. Denoting this image again by m_j, $Z_b(\lambda)$ has basis $m_0, \dots, m_{\ell-1}$ such that:

$$K m_i = q^{-2i} \lambda m_i$$

$$F m_i = \begin{cases} m_{i+1} & \text{if } i<\ell-1, \\ b m_0 & \text{if } i=\ell-1, \end{cases}$$

$$E m_i = \begin{cases} 0 & \text{if } i=0, \\ [i]\cdot \dfrac{\lambda q^{1-i} - \lambda^{-1} q^{i-1}}{q-q^{-1}} m_{i-1} & \text{if } i>0. \end{cases}$$

Since q is a primitive ℓ-th root of unity with ℓ odd, the $q^{-2i}\cdot\lambda$ for $0 \le i < \ell$ are distinct. Hence $Z_b(\lambda)_{q^{-2i}\cdot\lambda} = k m_i$ for $0 \le i < \ell$. In particular K can have eigenvalues other than $\pm q^a$ for $a \in \mathbb{Z}$ by choosing λ distinct from these.
(*)

We have that $F^\ell m_i = b m_i$ for $0 \le i < \ell$ and thus F^ℓ acts by multiplication by b on $Z_b(\lambda)$. By choosing $b \neq 0$ we have that F does not act nilpotently on the finite dimensional U-module $Z_b(\lambda)$.
(Δ)

Proposition: Suppose that q is not a root of unity, let M be a finite dimensional U-module. Then there are integers $r, s > 0$ with $E^r M = 0$ and $F^s M = 0$. (Δ) This will not happen.

Proposition: Suppose that q is not a root of unity and that $\operatorname{char}(k) \neq 2$, let M be a finite dimensional U-module. Then M is the direct sum of its weight spaces, and all weights of M have the form $\pm q^a$ with $a \in \mathbb{Z}$.

(*) This will not generalize.

Conceptual facts

1) A representation of an algebra is the same thing as a module over the algebra.
2) The universal enveloping algebra preserves the representation theory: the representations of a Lie algebra $\mathfrak{g}$ correspond to the modules over $U(\mathfrak{g})$.
   In fact, the category of all representations of $\mathfrak{g}$ is isomorphic to the category of left modules over $U(\mathfrak{g})$, as abelian categories.