20201006

Tuesday, 6 October 2020 05:25

Definition: Suppose that q is a primitive ℓ-root of unity with ℓ-odd, $\ell \geq 3$. Set for $b, \lambda \in k$, $\lambda \neq 0$:

$$Z_b(\lambda) := \frac{M(\lambda)}{U(m_\ell - b m_0)}.$$

It has basis $m_0, \dots, m_{\ell-1}$ such that:

$$K m_i = q^{-2i} \lambda m_i$$

$$F m_i = \begin{cases} m_{i+1} & \text{if } i < \ell-1, \\ b m_0 & \text{if } i = \ell-1, \end{cases}$$

$$E m_i = \begin{cases} 0 & \text{if } i=0, \\ [i] \cdot \dfrac{\lambda q^{1-i} - \lambda^{-1} q^{i-1}}{q - q^{-1}} m_{i-1} & \text{if } i > 0. \end{cases}$$

Moreover the $q^{-2i} \cdot \lambda$ for $0 \leq i < \ell$ are distinct, hence: $Z_b(\lambda)_{q^{-2i}\cdot\lambda} = k m_i$ for $0 \leq i < \ell$

Proposition: Suppose that q is a primitive ℓ-th root of unity with $\ell \geq 3$ odd

(i) If $b \neq 0$ or $\lambda^{2\ell} \neq 1$ then $Z_b(\lambda)$ is a simple U-module.

(ii) If $b = 0$ and $\lambda = \pm q^n$ with $0 \leq n < \ell$ then $Z_b(\lambda)$ is simple if and only if $n = \ell - 1$, for $n < \ell - 1$ the m_j with $j > n$ span a submodule of $Z_b(\lambda)$, and this is the only submodule different from 0 and $Z_b(\lambda)$

Proof: Let M be a nonzero submodule of $Z_b(\lambda)$. Since it is K-stable, M is the direct sum of its weight spaces: $M = \bigoplus_\mu M \cap Z_b(\lambda)_\mu$. Since $Z_b(\lambda)_{q^{-2i}\cdot\lambda} = k m_i$ for $0 \leq i < \ell$, M is spanned by the m_i contained in M Since $M \neq 0$, there is some $m_i \in M$ Choose $j \geq 0$ minimal with $m_j \in M$, then:

$m_i = F^{i-j} m_j \in M$ for all $j \leq i < \ell$ If $j = 0$ then $M = Z_b(\lambda)$.

Assume that $j > 0$. If $b \neq 0$ then $m_0 = b^{-1} F m_{\ell-1} \in M$ so $j = 0$, hence $b = 0$. Now M is the span of all m_i with $i \geq j$, and since $E m_j \in M$ is a multiple of $m_{j-1} \notin M$ we have $E m_j = 0$, hence $[j](\lambda q^{1-j} - \lambda^{-1} q^{j-1}) = 0$. Since $0 < j < \ell$ our assumption on q implies $[j] \neq 0$. Therefore $\lambda^2 = q^{2(j-1)}$ so $\lambda^{2\ell} = 1$ and $\lambda = \pm q^{(j-1)}$

When $b = 0$ and $\lambda = \pm q^n$ with $0 \leq n < \ell - 1$ the span of m_i for $i > n$ is a submodule as a consequence of $E m_{n+1} = 0$. $\square$.

Remark: The $Z_0(\pm q^n)$ with $0 \leq n < \ell - 1$ are not semisimple: the submodule spanned by m_i, $i > n$, has no complement. We found a finite dimensional U-module that is the direct sum of its weight spaces but is not semisimple. Hence another result for q not a root of unity won't be extended.

$\boxed{\text{Finite dimensional simple } U\text{-modules:}}$ when q is a primitive ℓ-th root of unity with $\ell \geq 3$ odd and $\bar{k} = k$.

(this guarantees that M is the direct sum of its weight spaces, and by Schur's Lemma $E^\ell, F^\ell, K^\ell, C$ act as scalars)

Case 1: E^ℓ acts as 0 on M.

Then $\{m \in M \mid Em = 0\} \neq 0$ and it is K-stable since $KEK^{-1} = q^2 E$. Hence it has an eigenvector of K. We can then find $m \in M$ with $m \neq 0$ and $\lambda \in k$ with $\lambda \neq 0$ such that $Em = 0$ and $Km = \lambda m$. By the universal property, there is a homomorphism $\varphi: M(\lambda) \longrightarrow M$, $m_0 \longmapsto m$. Since M is simple, φ is surjective. There is a scalar $b \in k$ such that F^ℓ acts as multiplication by b on M, so:

$$\varphi(F^\ell m_\ell - b m_0) = \varphi(F^\ell m_0) - bm = F^\ell m - bm = 0$$

Hence $U(m_\ell - bm_0)$ is contained in the kernel of φ so φ factors through $Z_b(\lambda)$. By the Proposition above, M is either isomorphic to $Z_b(\lambda)$ or $L(n, \pm)$.

<u>Case 2:</u> <u>F^ℓ acts as 0 on M and E^ℓ does not</u>

We use the automorphism
$$\omega: U \longrightarrow U, \quad E \longmapsto F, \quad F \longmapsto E, \quad K \longmapsto K^{-1}.$$
For any U-module N, set ${}^\omega N$ to be the U-module equal to N as a vector space and where each $u \in U$ acts on ${}^\omega N$ as $\omega(u)$ acts on N. Clearly ${}^\omega({}^\omega N) \cong N$ and ${}^\omega N$ is simple if and only if N is simple.

Now ${}^\omega M$ is a simple module as in Case 1 with $b \neq 0$, so M is isomorphic to some ${}^\omega Z_b(\lambda)$. It has dimension ℓ and there is a basis $m_0, \dots, m_{\ell-1}$ such that the action of U is:

$$K m_i = q^{+2i} \lambda m_i$$

$$F m_i = \begin{cases} 0 & \text{if } i=0, \\ [i] \cdot \dfrac{\lambda q^{1-i} - \lambda^{-1} q^{i-1}}{q - q^{-1}} m_{i-1} & \text{if } i>0, \end{cases}$$

$$E m_i = \begin{cases} m_{i+1} & \text{if } i<\ell-1, \\ b m_0 & \text{if } i=\ell-1. \end{cases}$$

<u>Case 3:</u> <u>F^ℓ and E^ℓ do not act as 0 on M.</u>

Let $b \in k, b \neq 0$, the scalar through which F^ℓ acts on M. Let $m_0 \neq 0$ be an eigenvector of eigenvalue λ for K. Set: $m_i = F^i m_0$ for $0 < i < \ell$, since $F^{\ell-i} m_i = F^\ell m_0 = bm_0 \neq 0$ we have $m_i \neq 0$.

Now K and F act as:

$$K m_i = q^{-2i} \lambda m_i,$$

$$F m_i = \begin{cases} m_{i+1} & \text{if } i<\ell-1, \\ b m_0 & \text{if } i=\ell-1 \end{cases}$$

Since the $q^{-2i}\lambda$ with $0 \leq i < \ell$ are distinct, the m_i are linearly independent (eigenvectors corresponding to different eigenvalues).

The central element C acts through a scalar on M, and since:

$$FE m_0 = C m_0 - \frac{Kq + K^{-1} q^{-1}}{(q-q^{-1})^2} m_0,$$ there is an $a' \in k$ with $FE m_0 = a' m_0$.

Thus: $b E m_0 = F^\ell E m_0 = F^{\ell-1} a' m_0 = a' m_{\ell-1}$, so: $E m_0 = a m_{\ell-1}$ with $a = \frac{a'}{b}$. Now:

$$E m_i = EF^i m_0 = F^i E m_0 + [i] F^{i-1} [K; 1-i] m_0$$ for all $i > 0$, hence:

$$E m_i = \begin{cases} a\, m_{\ell-1} & \text{if } i=0, \\ \left(ab + \dfrac{(q^i - q^{-i})(\lambda q^{1-i} - \lambda^{-1} q^{i-1})}{(q-q^{-1})^2} \right) m_{i-1} & \text{if } i>0 \end{cases}$$

We then have that the span of the m_i is stable under all generators of U, hence equal to the simple module M. Hence the m_i are a basis of M and the above actions describe the module completely.

and the above actions describe the module completely.

Note: Given a, b, λ we can use the actions above to define a U-module. If $b \neq 0$ then this module is simple (the proof of the first Proposition holds). If a and all $ab + (q^i + q^{-i})(\lambda q^{1-i} - \lambda^{-1} q^{i-1})(q - q^{-1})^{-2}$ are not zero, then E^ℓ does not act as zero and we are in Case 3 above.

The module M uniquely determines b, but does not determine a nor λ: we could choose instead of m_0 any other m_i. We can thus replace λ by $q^{-2i}\lambda$ and a by $a + (q^i + q^{-i})(\lambda q^{1-i} - \lambda^{-1} q^{i-1})(q - q^{-1})^{-2} b^{-1}$ and get an isomorphic module

$\boxed{\text{Goal: Determine the center of } U}$

Recall: U is graded: $U = \bigoplus_{m \in \mathbb{Z}} U_m$ where U_m is spanned by all $F^s K^n E^r$ with $m = r - s$

Lemma: a) If q is not a root of unity, then the center of U is contained in U_0.

b) If q is a primitive ℓ-th root of unity with $\ell \geq 3$ odd, then the center of U is generated by E^ℓ, F^ℓ, and their intersection with U_0.

Proof: The center of a graded algebra inherits the grading. We need to find for each $m \in \mathbb{Z}$ which elements of U_m are central. If $0 \neq u \in U_m$ is central then: $KuK^{-1} = q^{2i}u$ implies $q^{2m} = 1$.

If q is not a root of unity, this means $m = 0$.

If q is a primitive ℓ-th root of unity with $\ell \geq 3$ odd, this means $m = a\ell$ for some $a > 0$. Then U_m is spanned by all $F^s K^n E^{s+a\ell}$, so any $u \in U_m$ can be decomposed as: $u = u' E^{a\ell}$ with $u' \in U_0$. Then u is central if and only if u' is central (since E^ℓ is central, hence $E^{a\ell}$ is central, and U has no zero divisors). Similarly, if $m = -a\ell$ for some $a > 0$: $u = F^{a\ell} u'$ with $u' \in U_0$ central. $\square$

Recall: Elements $u \in U_0$ can be written uniquely as: $u = \sum_{r \in \mathbb{N}} F^r h_r E^r$ with almost all $h_r \in U_0$ zero.

Lemma: Let $u = \sum_{r \in \mathbb{N}} F^r h_r E^r \in U_0$, then it is central in U if and only if:

$$h_r - \gamma_{-2}(h_r) = [r+1][K;-r]h_{r+1} \quad \text{for all } r \geq 0$$

Proof: We have:

$$Eu = \sum_{r \in \mathbb{N}} E F^r h_r E^r = \sum_{r \in \mathbb{N}} F^r E h_r E^r + \sum_{r \in \mathbb{N}^+} [r] F^{r-1} [K; 1-r] h_r E^r =$$

$$= \sum_{r \in \mathbb{N}} F^r \gamma_{-2}(h_r) E^{r+1} + \sum_{r \in \mathbb{N}} [r+1] F^r [K;-r] h_{r+1} E^{r+1}.$$

$$uE = \sum_{r \in \mathbb{N}} F^r h_r E^{r+1}$$

So $Eu = uE$ if and only if the desired equality is satisfied. The same condition suffices to show $Fu = uF$, and $Ku = uK$ always holds for any $u \in U_0$. $\square$.