Definition: Let k be a field, fix $q \in k$ with $q \neq 0$, $q^2 \neq 1$. Then $U_q(sl_2)$ is the associative unital algebra over k generated by E, F, K, K^{-1} with relations

(R1) $KK^{-1} = 1 = K^{-1}K$.

(R2) $KEK^{-1} = q^2 E$

(R3) $KFK^{-1} = q^{-2}F$.

(R4) $EF - FE = \dfrac{K - K^{-1}}{q - q^{-1}}$

$$C = FE + \frac{Kq + K^{-1}q^{-1}}{(q-q^{-1})^2}.$$

We may abuse $U := U_q(sl_2)$.

Lemma: a) If q is not a root of unity, then the center of U is contained in U_0.

b) If q is a primitive ℓ-th root of unity with $\ell \geq 3$ odd, then the center of U is generated by E^ℓ, F^ℓ, and their intersection with U_0.

Lemma: Let $n = \sum_{r \in \mathbb{N}} F^r h_r E^r \in U_0$, then it is central in U if and only if:

$$h_r - \gamma_{-2}(h_r) = [r+1][K; -r] h_{r+1} \quad \text{for all } r \geq 0$$

Definition: Set
$$\pi : U_0 \longrightarrow U^0.$$
$$\sum_{r \in \mathbb{N}} F^r h_r E^r \mapsto h_0$$

Remark: We have $\ker(\pi) = F U_0 E$, and π is an algebra homomorphism since the kernel is a two sided ideal.

Lemma: If q is not a root of unity, then π induces an injective homomorphism from $Z(U)$ to U^0.

Proof. The Lemma above says that h_{r+1} is determined by h_r, since q not a root of unity implies $[r+1][K; -r] \neq 0$, and since U^0 is an integral domain. Hence all h_r are inductively determined by $h_0 = \pi(n)$, proving injectivity. $\square$.

Lemma: Suppose that q is not a root of unity. Let $n \in U$ be central and write $\gamma_{-1} \circ \pi(n) = \sum_{i \in \mathbb{Z}} a_i K^i$ with $a_i \in k$ almost all zero. Then $a_i = a_{-i}$ for all $i \in \mathbb{Z}$.

Proof: We have $\pi(n) = \sum_{i \in \mathbb{Z}} a_i q^i K^i$, hence: $n = \sum_{i \in \mathbb{Z}} a_i q^i K^i + \sum_{r > 0} F^r h_r E^r$ for suitable $h_r \in U^0$.

Then n acts on $M(\lambda)$ as multiplication by $\sum_{i \in \mathbb{Z}} a_i q^i \lambda^i$, since it acts like that on the generator m_0.

In particular, n acts as multiplication by $\sum_{i \in \mathbb{Z}} a_i q^{(n+1)i}$ on $M(q^n)$ for any $n \in \mathbb{Z}$. We saw that for $n \geq 0$ there is a submodule isomorphic to $M(q^{-n-2})$ in $M(q^n)$, so since n has to act by the same scalar on both modules.

$$\sum_{i \in \mathbb{Z}} a_i q^{(n+1)i} = \sum_{i \in \mathbb{Z}} a_i q^{-(n+1)i}, \text{ meaning } \sum_{i \in \mathbb{Z}} a_i q^{ni} = \sum_{i \in \mathbb{Z}} a_i q^{-ni} = \sum_{i \in \mathbb{Z}} a_{-i} q^{ni}$$

for all $n \in \mathbb{Z}$. Denote by $\psi_i : \mathbb{Z} \longrightarrow k^\times$ the group homomorphism from $\mathbb{Z}$ to the multiplic group of k. $\quad n \longmapsto q^{ni}$ ... the multiplicative

Since $\psi_i(1) = q^i$ and q is not a root of unity, the ψ_i are all distinct for $i \in \mathbb{Z}$. Hence the ψ_i are linearly independent over k by Artin's theorem on the linear independence of characters. The above equation is

$\sum_{i \in \mathbb{Z}} (a_i - a_{-i}) \psi_i = 0$, so by linear independence we have $a_i - a_{-i} = 0$ for all $i \in \mathbb{Z}$. $\square$.

Proposition: Suppose that q is not a root of unity. Then the center of U is generated by C as a k-algebra. It is isomorphic to a polynomial ring over k in one indeterminate.

Proof. Let $s: U^0 \longrightarrow U^0$, $K \longmapsto K^{-1}$, namely $s(K^i) = K^{-i}$ for all $i \in \mathbb{Z}$. The Lemma above says that $\gamma_{-1} \circ \pi$ maps the center of U to the subalgebra $(U^0)^s = \{h \in U^0 \mid s(h) = h\}$ of fixed points of s in U^0. Clearly $K^n + K^{-n}$ for $n > 0$ with 1 form a basis of $(U^0)^s$, so $(K + K^{-1})^n$ for $n \geq 0$ are also a basis. Since $\gamma_{-1} \circ \pi(1) = 1$ and $\gamma_{-1} \circ \pi(C) = (q - q^{-1})^{-2}(K + K^{-1})$, we have that $\gamma_{-1} \circ \pi$ maps $k[C]$ surjectively to $(U^0)^s$. Since $\gamma_{-1} \circ \pi$ is injective on $Z(U)$, then $\gamma_{-1} \circ \pi$ induces an isomorphism between $Z(U)$ and $(U^0)^s$, which is equal to $k[C]$. It is isomorphic to a polynomial ring since the $\gamma_{-1} \circ \pi(C^n)$ are linearly independent. $\square$.

Definition: Suppose that q is a primitive ℓ-root of unity, $\ell \geq 3$ odd. Denote:

$$U_0' = \left\{ \sum_{r=0}^{\ell-1} F^r h_r E^r \;\middle|\; h_0, \dots, h_{\ell-1} \in U^0 \right\} \subseteq U_0$$

Lemma: a) The k-algebra $Z(U)$ is generated by E^ℓ, F^ℓ, and $Z(U) \cap U_0'$.

b) The restriction of π to $Z(U) \cap U_0'$ is injective.

Proof: a) We proved that the center of U is generated by E^ℓ, F^ℓ, and its intersection with U_0. It is enough to look at $Z(U) \cap U_0$. Any $u \in U_0$ can be written $u = \sum_{r \geq 0} F^r h_r E^r$ with $h_r \in U^0$ almost all zero. Then:

$$u = \sum_{j \geq 0} F^{j\ell} u_j E^{j\ell} \quad \text{where} \quad u_j = \sum_{r=0}^{\ell-1} F^r h_{j\ell + r} E^r \in U_0'.$$

Since $[j\ell] = 0$ for all j:

$$Eu = \sum_{j \geq 0} EF^{j\ell} u_j E^{j\ell} = \sum_{j \geq 0} F^{j\ell} E u_j E^{j\ell} + \sum_{j > 0} [j\ell] F^{j\ell - 1} [K; 1-r] h_r E^r = \sum_{j \geq 0} F^{j\ell} E u_j E^{j\ell}$$

and similarly for Fu. Since $uK = Ku$ for all $u \in U_0$, we have that u is central if and only if all u_j are central.

b) By one of the Lemmas above: $u_r - \gamma_{-2}(u_r) = [r+1][K; -r] u_{r+1}$ for all $r \geq 0$, and since $[r] \neq 0$ for $0 < r < \ell$, the result follows. $\square$.

Proposition: Suppose that q is a primitive ℓ-th root of unity, $\ell \geq 3$ odd. Then the center of U is generated by E^ℓ, F^ℓ, K^ℓ, $K^{-\ell}$, and C.

Proof: By the Lemma above, it is enough to show that $Z(U)\cap U_0' \subseteq k[K^\ell, K^{-\ell}, C]$. All $K^{j\ell}C^r$ with $j\in\mathbb{Z}$, $0\le r<\ell$ are contained in $Z(U)\cap U_0'$. We have:

$$\gamma_{-1}\circ\pi\Big(\sum_{j\in\mathbb{Z}}\sum_{r=0}^{\ell-1} kK^{j\ell}C^r\Big) = \sum_{j\in\mathbb{Z}}\sum_{r=0}^{\ell-1} kK^{j\ell}(K+K^{-1})^r \subseteq \gamma_{-1}\circ\pi(Z(U)\cap U_0').$$

By part b) of the above Lemma, it is enough to show that we have an equality above. First, notice that all $K^{j\ell}(K+K^{-1})^r$ with $j\in\mathbb{Z}$, $0\le r<\ell$, and K^s for $0<s<\ell$, form a basis of U^0. Suppose that the above inclusion is strict, then $\gamma_{-1}\circ\pi(Z(U)\cap U_0')$ has to intersect $\sum_{s=1}^{\ell-1} kK^s$ nontrivially. Take then $u\in Z(U)\cap U_0'$ with $\gamma_{-1}\circ\pi(u)\in\sum_{s=1}^{\ell-1} kK^s$, to prove equality above it is good enough to prove $u=0$. Write:

$$u=\sum_{r=0}^{\ell-1} F^r h_r E^r \quad\text{where}\quad h_r=\sum_{s\in\mathbb{Z}} a_{r,s}K^s \quad\text{with all } a_{r,s}\in K^s.$$

The assumption $\gamma_{-1}\circ\pi(u)\in\sum_{s=1}^{\ell-1} kK^s$ says that $a_{0,s}=0$ for all $s\le 0$ and all $s\ge\ell$. We check by induction on r that this vanishing holds for all $a_{r,s}$. Since u is central, by one of the Lemmas above:

$$h_r-\gamma_{-2}(h_r)=[r+1][K;-r]h_{r+1}, \quad \text{we will use this for } r+1<\ell.$$

We have on one side:

$$h_r-\gamma_{-2}(h_r)=\sum_{s\in\mathbb{Z}} a_{r,s}(1-q^{-2s})K^s$$

And on the other side:

$$(Kq^{-r}-K^{-1}q^{r})h_{r+1}=\sum_{s\in\mathbb{Z}}(a_{r+1,s-1}q^{-r}-a_{r+1,s+1}q^{r})K^s.$$

If n is the largest s with $a_{r+1,s}\neq 0$ then $a_{r+1,n}q^{-r}K^{n+1}$ is the top term on the right hand side of the above equality. If m is the smallest s with $a_{r+1,s}\neq 0$ then $a_{r+1,m}q^{r}K^{m-1}$ is the bottom term on the right hand side of the above equality. Then $a_{r,n+1}\neq 0$ and $a_{r,m-1}\neq 0$. By induction we get $n+1<\ell$ and $m-1>0$, hence $n<\ell$ and $m>0$.

We now want to use induction on r from above to show that $h_r=0$ for all r. For $r=\ell-1$ we get $h_r-\gamma_{-2}(h_r)=0$ since $[r+1]=[\ell]=0$. By induction, we get this for all $r<\ell-1$. Then:

$$h_r-\gamma_{-2}(h_r)=\sum_{s\in\mathbb{Z}} a_{r,s}(1-q^{-2s})K^s \quad\text{implies that } a_{r,s}=0 \text{ or } 1=q^{-2s} \text{ for all } s\in\mathbb{Z}.$$

Since q is a primitive ℓ-th root of unity, ℓ odd, we have $q^{-2s}\neq 1$ for $0<s<\ell$, hence $a_{r,s}=0$ for these s. All other $a_{r,s}$ were already equal to 0, so indeed $h_r=0$. $\square$

Remark: We have seen that E^ℓ, F^ℓ, and K^ℓ are algebraically independent over k by the PBW-type basis. One can show that C is integral over the subalgebra generated by $E^\ell, F^\ell, K^\ell, K^{-\ell}$.