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3. Representation theory of quantized enveloping algebras.

Idea: "Quantization" replaces commutative algebras with non-commutative ones.

This can be thought as a deformation of the algebra.

Examples:

(1) The quantum plane (or skew polynomial ring):

$$k_q[x,y] := k[x,y]/(xy = qyx) \quad \text{for some } q \in k^*$$

is a quantization of $k[x,y]$.

(2) The Weyl algebra:

$$W(2) := k[x,y]/(xy - yx = 1)$$

is a quantization of the symmetric algebra $k[x,y]$.

It is also an Ore extension of $k[y]$ via $\frac{\partial}{\partial y} =: \delta \in \mathrm{Der}(k[y])$.

(3) $\mathcal{U}_q(sl_2)$ is a quantization of $\mathcal{U}(sl_2)$.

The formal parameter q is introduced in the Serre relations.

Theorem [Serre]: If $A \in M_n(k)$ is a Cartan matrix, then up to isomorphism there is a unique semisimple complex Lie algebra $\mathfrak{g}$ with Cartan matrix equivalent to A. It is defined by generators:

$\{e_i, f_i, h_i\}_{i=1}^n$ and relations

S1. $[h_i, h_j] = 0$

S2. $[e_i, f_j] = h_i \cdot \delta_{ij}$

S3. $[h_i, e_j] = A_{ij} e_j$

S4. $[h_i, f_j] = -A_{ij} f_j$

S5. $\mathrm{ad}_{e_i}^{1-A_{ij}}(e_j) = 0$

S6. $\mathrm{ad}_{f_i}^{1-A_{ij}}(f_j) = 0$.

These relations prescribe how $\mathfrak{g}$ is supposed to decompose when considered as a module over $sl_2(i) := \langle e_i, f_i, h_i \rangle_{\mathfrak{g}} \subseteq \mathfrak{g}$.

With care, q can be specialized to nonzero complex numbers.

3.1. Weyl modules.

- For $q \neq 0$ not a root of unity, the category of finite-dimensional $U_q(\mathfrak{g})$-modules with weight decomposition:

$$V = \bigoplus_{\lambda \in P} V^\lambda \,, \quad q^h|_{V^\lambda} = q^{(h,\lambda)} \, id_{V^\lambda} \,, \quad e_i^{(n)}(V^\lambda) \subset V^{\lambda + n\alpha_i} \,, \quad f_i^{(n)}(V^\lambda) \subset V^{\lambda - n\alpha_i}$$

  has the same isomorphism classes of simple objects and fusion coefficients as the category of finite-dimensional $U(\mathfrak{g})$-modules.

  In turn, this is essentially the category of finite-dimensional representations of $\mathfrak{g}$.

- For q a root of unity, the category of finite-dimensional $U_q(\mathfrak{g})$-modules with weight decomposition:

$$V = \bigoplus_{\lambda \in P} V^\lambda \,, \quad q^h|_{V^\lambda} = q^{(h,\lambda)} \, id_{V^\lambda} \,, \quad e_i^{(n)}(V^\lambda) \subset V^{\lambda + n\alpha_i} \,, \quad f_i^{(n)}(V^\lambda) \subset V^{\lambda - n\alpha_i}$$

  is a ribbon category.

  However, it is not semisimple and it has infinitely many isomorphism classes of simple objects.

- To avoid the technicalities of the weight decomposition, we will consider "big enough" q.
- Let m be the largest absolute value of an off-diagonal entry of the Cartan matrix of $\mathfrak{g}$.

A, D, E	$m = 1$	(simply laced)
B, C, F	$m = 2$	
G	$m = 3$	

- Let q be a root of unity and $\mathrm{ord}(q^2) = \ell$.

  If $\mathfrak{g}$ is a semisimple Lie algebra with $m \mid \ell$, we say that q is divisible for $\mathfrak{g}$.

  The representation theory of $U_q(\mathfrak{g})$ will depend on ℓ or on q and ℓ.

- Lower bound for ℓ producing a modular tensor category (in future sections).

Type	Divisible	$\ell \geq$
A_n	✓	$n+1$
B_n	✓	$2n+1$

[illegible]	×	[illegible]
	✓	$4n-2$
C_n	×	$2n+1$
	✓	$2n+2$
D_n	✓	$2n-2$
E_6	✓	12
E_7	✓	18
E_8	✓	30
F_4	×	13
	✓	18
G_2	×	7
	✓	12

- To construct a semisimple category, we first consider Weyl modules labeled by $\lambda \in Q$. Q: weight lattice.

Def: $\lambda \in \Lambda^+$ a dominant integer weight of $\mathfrak{g}$, the Weyl module V_λ of $U_q(\mathfrak{g})$ is:

$$V_\lambda := (V_\lambda)_{\mathbb{Z}} \otimes_{\mathcal{A}} \mathbb{C}$$

for $\mathcal{A} = \mathbb{Z}[q^{\pm 1/|\Lambda/Q|}]$, $(V_\lambda)_{\mathbb{Z}} := U_q(\mathfrak{g})_{\mathbb{Z}} v_\lambda$ the $U_q(\mathfrak{g})_{\mathbb{Z}}$-submodule generated by the highest weight vector.

- The quantum dimension is:

$$\dim(V_\lambda) = \prod_{\alpha > 0} \frac{[\langle \alpha, \lambda + \rho \rangle]}{[\langle \alpha, \rho \rangle]} \qquad \text{where } [n] = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

Quantum Weyl Formula — Quantum integer.

Example: The classical formula for the dimension of the representation of highest weight $s\lambda_1 + t\lambda_2$ of G_2 is:

$$\frac{1}{5!}(s+1)(t+1)(s+t+2)(s+2t+3)(s+3t+4)(2s+3t+5)$$

And the quantum formula is:

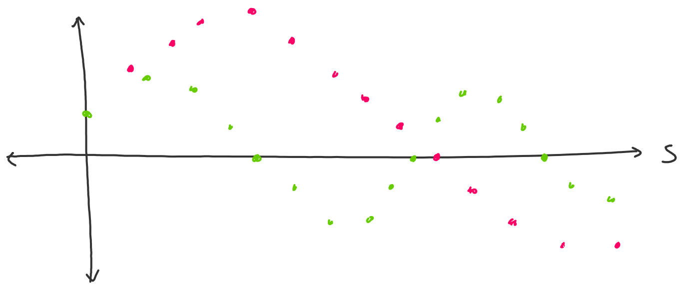
$$\frac{[s+1][3(t+1)][3(s+t+2)][3(s+2t+3)][s+3t+4][2s+3t+5]}{[1][3][6][9][4][5]}.$$

Example: Dimensions of $\mathfrak{sl}_2$.

We have Dynkin diagram: $\bullet$, so exactly one positive root α and one fundamental weight λ. By the above, the Weyl module of weight $s\lambda$ has:

$$\dim(V_{s\lambda}) = [s+1].$$

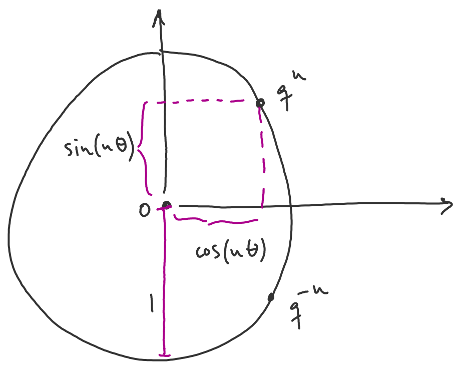
For $q = e^{\frac{\pi i}{10}}$ in red and $q = e^{\frac{2\pi i}{9}}$ in green we have dimensions:



Which sometimes are zero.

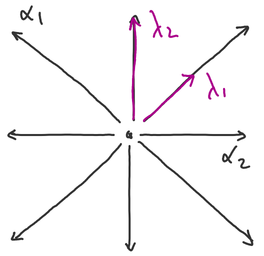
This is an inherently trigonometric problem.

When $q = e^{i\theta}$ we have $q^n - q^{-n} = 2i\sin(n\theta)$ and $q^n + q^{-n} = 2\cos(n\theta)$.



Example: Dimensions for B_2.

We have Dynkin diagram:



, if $q = e^{i\theta}$ with big enough order, then for $\lambda = s\lambda_1 + t\lambda_2$:

$\lambda_1 = \frac{\alpha_1}{2} + \alpha_2$, $\lambda_2 = \alpha_1 + \alpha_2$

$$\dim(V_{s\lambda_1 + t\lambda_2}) = \frac{\sin((s+1)\theta)\cdot\sin(2(t+1)\theta)\cdot\sin(2(s+t+2)\theta)\cdot\sin((s+2t+3)\theta)}{\sin(\theta)\cdot\sin(2\theta)\cdot\sin(3\theta)\cdot\sin(4\theta)}$$

3.2. Affine Weyl group.

- Def: The affine Weyl group W is the group generated by the reflections T_i corresponding to hyperplanes:

$$T_i := \{\lambda \in \mathfrak{t}^* : \langle \lambda + \rho, \alpha_i^\vee \rangle = 0\}$$

for all simple roots $\alpha_i \in \Delta$, and the single reflection through the hyperplane:

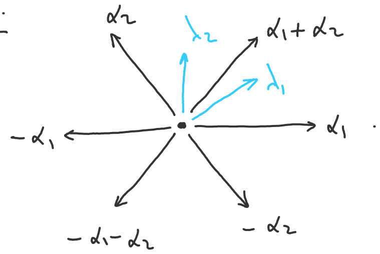
$$T_0 := \{ \lambda \in \mathfrak{t}^* : \langle \lambda + \rho, \beta^\vee \rangle = \ell \}$$

for β the longest root if $m \mid \ell$, and the shortest root if $m \nmid \ell$.

- The T_i with $i > 0$ are the reflections generating the classical Weyl group shifted by $-\rho$.
- The weights $\lambda \in Q$ strictly bounded by T_i will be called the Weyl alcove, denoted Λ_0.
- The planes T_i are called walls of the Weyl alcove.

Example: Affine Weyl group of $\mathfrak{sl}_3$.

We have Dynkin diagram:



Recall that for $\mathfrak{sl}_n$ with $h_1, \dots, h_r$ a basis for $\mathfrak{t} \subseteq \mathfrak{sl}_n$, a spanning set for $\mathfrak{t}^*$ are the functionals $\varepsilon_1, \dots, \varepsilon_n \in \mathfrak{t}^*$ with $\varepsilon_i(h_j)$ is the i-th diagonal entry of h_j.

Now:

$\Delta = \{ \alpha_1, \dots, \alpha_{n-1} \}$ with $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ are the simple roots.

They form a basis of Φ the root system, generating $P = \mathbb{Z}\Phi$ the root lattice.

The fundamental weights are:

$$\lambda_r = \sum_{j=1}^{r} \varepsilon_j - \frac{r}{n} \sum_{j=1}^{n} \varepsilon_j \qquad \text{for } 1 \leq r \leq n-1.$$

Hence for $\mathfrak{sl}_3$ we have two fundamental weights λ_1, λ_2, with:

$$\lambda_1 = \varepsilon_1 - \frac{1}{3}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) = \frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_2,$$

$$\lambda_2 = \varepsilon_1 + \varepsilon_2 - \frac{2}{3}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) = \frac{1}{3}\alpha_1 + \frac{2}{3}\alpha_2.$$

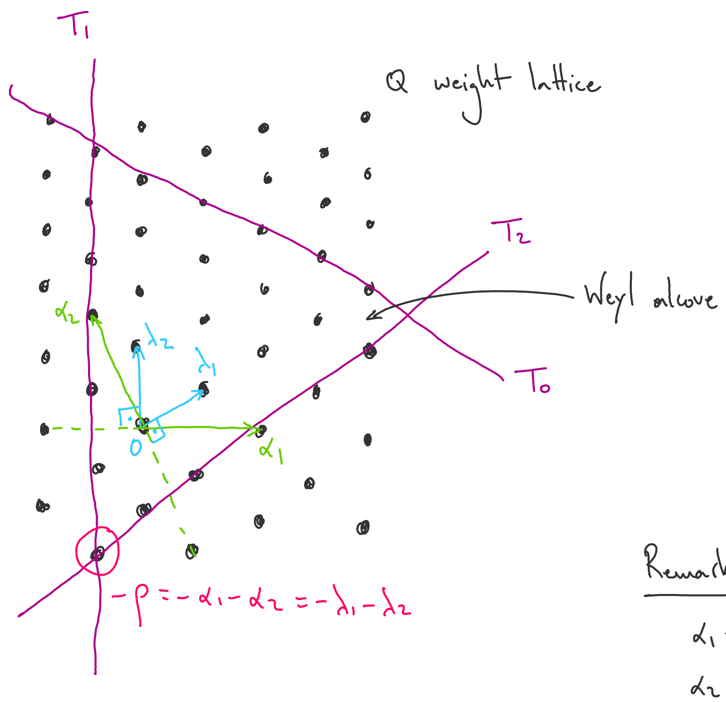
and: $\langle \lambda_1, \lambda_1 \rangle = \langle \lambda_2, \lambda_2 \rangle = \frac{2}{3}$, $\langle \lambda_1, \lambda_2 \rangle = \frac{1}{3}$.

Now $\lambda = s\lambda_1 + t\lambda_2$ lies on T_1 if and only if $s = -1$.

T_2 if and only if $t = -1$.

T_0 if and only if $s + t = \ell - 2$.

Hence for $\ell = 6$ this looks like:



Remark:

$\alpha_1 = 2\lambda_1 - \lambda_2$,

$\alpha_2 = -\lambda_1 + 2\lambda_2$.

Remark:

In general, the affine Weyl group $\mathcal{W}$ is an affine Coxeter group. These are classified by Dynkin diagrams in a similar fashion as simple finite-dimensional Lie algebras.

- There is a connection between the representation theory of affine Lie algebras and the representation theory of quantum groups at roots of unity:

Fix $\mathfrak{g}$ a simple complex Lie algebra, let $\mathcal{O}_k^{int}$ be the category of integrable modules of level $k \in \mathbb{Z}_+$ over the corresponding affine Lie algebra $\hat{\mathfrak{g}}$.

Let $\mathcal{C}^{int}(\mathfrak{g}, x)$ be a certain subquotient of the category of representations of the quantum group $\mathcal{U}_q(\mathfrak{g})$ for $q = e^{\frac{\pi i}{m x}}$:

Objects: tilting modules.

Morphisms: $\mathrm{Hom}_{\mathcal{C}^{int}(\mathfrak{g},x)}(V,W) = \mathrm{Hom}_{\mathcal{T}}(V,W) / \text{negligible morphisms}$

Where $\mathcal{T}$ is the category of tilting modules.

Theorem:

The is an equivalence of modular tensor categories: $\mathcal{O}_k^{int} \simeq \mathcal{C}^{int}(\mathfrak{g}, x)$ for $x = k + h^\vee$ where $h^\vee$ is the dual Coxeter number for $\mathfrak{g}$.

- The affine Weyl group acts anti-symmetrically on the dimensions of the Weyl modules: if μ is $\mathcal{W}$-conjugate to $\lambda \in \Lambda_0$ by $\tau \in \mathcal{W}$, then $\dim(V_\mu) = (-1)^{\ell(\tau)} \cdot \dim(V_\lambda)$ where $\ell(\tau)$ is the length of a shortest expression of τ in terms of the generating reflections τ_i.
- The quantum dimensions of Weyl modules with highest weights $\lambda \in \Lambda_0$ determine the quantum dimensions of all other Weyl modules.
- If λ lies on any hyperplane of reflection arising from $\mathcal{W}$, then $\dim(V_\lambda) = 0$. The converse is also true.

Example: Affine Weyl group of sl_2.

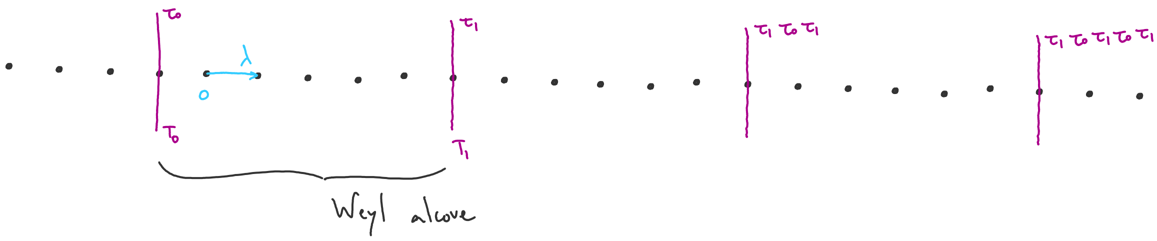
As before, the weight lattice is the $\mathbb{Z}$-linear span of the unique fundamental weight λ with $\langle \lambda, \lambda \rangle = \frac{1}{2}$. If for some $s \in \mathbb{Z}_{\geq 0}$ and $\ell \in \mathbb{Z}_{\geq 2}$ we have:

$\langle (s+1)\lambda, 2\lambda \rangle = \ell$, then $s = \ell - 1$.

$\langle (s+1)\lambda, 2\lambda \rangle = 0$, then $s = -1$.

Then the affine Weyl group is the reflections through $-1 + j\ell$ for $j \in \mathbb{Z}$.

Hence for $\ell = 6$ this looks like:



The anti-symmetric action can be visualized in the example about the dimensions of sl_2-modules.