Rep Theory

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1. What is a representation?

Often we try to linearize problems in order to make them easier to understand and in this way find approximate behaviour (calculus, Maclaurin series, tangent spaces) and tools (linear algebra) to study it.
Groups are not linear in general. But we can try to make them do linear things.

group

Definition 1.1 (Representation). *A (linear) representation* of a group G on a vector space V is a homomorphism

$$\rho : G \to GL(V).$$

If it is clear what G is intended, we write (ρ, V); if it is clear which ρ is intended, we write V. The $\boxed{\textit{degree}}$ of the representation is the dimension of $\boxed{V}$.

What does this mean?
Each $g \in G$ gives rise to an invertible endomorphism $\rho(g)$ of V which we think of as g acting (doing something) linearly on V, i.e., for any $v, v' \in V, \lambda \in F$ (field which V is over),

$$\begin{cases} \rho(g)(v+v') = \rho(g)(v) + \rho(g)(v') \\ \rho(g)(\lambda v) = \lambda \rho(g)(v) \\ \rho(e_G)(v) = v \\ \rho(gh)(v) = \rho(g)(\rho(h)(v)). \end{cases}$$

So via ρ, G gives some symmetries of V. Note that

$$\rho(e_G) = 1_V, \rho(gh) = \rho(g)\rho(h), \text{ and } \rho(g^{-1}) = \rho(g)^{-1}.$$

Thus, this notation captures much of how you thought about groups before you learned the actions of a group.

Example 1.2 (A representation of cyclic group of 4). $G = C_4 = \{e, x, x^2, x^3\}$, a cyclic group of order 4. Let V to be the ~~real plane~~. Let

$\mathbb{R}^2$

$x^j \to$

$$\rho : G \to GL(V)$$

given by a rotation clockwise by $j\pi/2$ radius.

Example 1.3 (Trivial representation). Any group G, $V = F$, $\rho(g) = 1_V$, identity operator. This is a representation of G, not surprising it is called the trivial representation of G.

Example 1.4 (Dihedral group). $G = D_n$ V be the plane, ρ the "natural representation", i.e., D_n is a symmetry group.

$\langle a, b : a^n = b^2 = e, bab = a^{-1} \rangle$

D_{2n}.

$\boxed{D_n}$

For example, if $n = 8$, you can rotate stop signs with 16 different ways.

$\pi = (\ )(\ )(\ )(\ )$

Example 1.5 (Symmetric group). $G = S_n$, $V = F$, for any $\pi \in S_n$, $\rho(\pi) = \begin{cases} +1_V & \text{if } \pi \text{ is even} \\ -1_V & \text{if } \pi \text{ is odd} \end{cases}$.

For example, if $n = 3$,

$(123) = (12)(13)$

$(132) = (12)(23)$

$$\rho(e) = \rho((123)) = \rho((132)) = 1_V$$
$$\rho((12)) = \rho((23)) = \rho((13)) = -1_V$$

We need to decide when the representations are indistinguishable.

Definition 1.6. Suppose (ρ_1, V_1) and (ρ_2, V_2) are representation of G. Then, an invertible linear transformation $\phi : V_1 \to V_2$ that *intertwines* the action of G, i.e.,

$$\boxed{\phi(\rho_1(g)(v))} = \boxed{\rho_2(g)(\phi(v))}, \quad \forall v \in V_1, g \in G$$

is called an *isomorphism*.
We write $\boxed{(\rho_1, V_1) \cong (\rho_2, V_2)}$ or $\boxed{V_1 \cong V_2}$.

We can write this more economically as

$$\boxed{\rho_2 = \phi \circ \rho_1 \circ \phi^{-1}}$$

which means $\boxed{\rho_2(g) = \phi \circ \rho_1(g) \circ \phi^{-1}}$ for all $g \in G$.

$v \longmapsto \phi(v)$

$$\begin{tikzcd} V_1 \arrow[r, "\phi", "\cong"'] \arrow[d, "\rho_1(g)"'] \arrow[dr, phantom, "\circlearrowright"] & V_2 \arrow[d, "\rho_2(g)"] \\ V_1 \arrow[r, "\cong", "\phi"'] & V_2 \end{tikzcd}$$

$\rho_2(g)(\phi(v))$

2. Basic problem: Describe all representation of a group up to isomorphism

Although the above definition of isomorphism is obvious, it is great because of what you know about vector spaces. Choosing a basis $\mathcal{B}$ for V produces a vector space isomorphism $V \cong F^n$ for some n. This then gives us a representation of G on F^n isomphism to (ρ, V) such that

$$\rho' : G \to GL(F^n) = GL_n(F)$$

where $\forall g \in G$, $\rho'(g)$ is the matrix version of $\rho(g)$ with respect to $\mathcal{B}$.

Example 2.1 (Continuation of Example 1.2). $G = C_4$, V is plane with the standard basis, i.e., $\{[\begin{smallmatrix}1\\0\end{smallmatrix}], [\begin{smallmatrix}0\\1\end{smallmatrix}]\}$. Then the isomorphic representaion is

$$e \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, x \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, x^2 \mapsto \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, x^3 \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Example 2.2 (Continuation of Example 1.3). For any G, V with basis r, the isomphic representation is identity matrix of dimension $\dim V$.

Example 2.3 (Continuation of Example 1.4). $G = D_3 = \langle a, b | a^3 = b^2 = e, bab^{-1} = a^{-1} \rangle$. Let V be a plane with the standard basis $\{[\begin{smallmatrix}1\\0\end{smallmatrix}], [\begin{smallmatrix}0\\1\end{smallmatrix}]\}$. Then, a is rotation counterclockwise by $2\pi/3$, b is a reflection in y-axis. Thus,

$$a \overset{\rho'}{\longmapsto} \begin{pmatrix} \cos(2\pi/3) & -\sin(2\pi/3) \\ \sin(2\pi/3) & \cos(2\pi/3) \end{pmatrix} = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$$
$$\boxed{b \overset{\rho'}{\longmapsto} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.}$$

ρ'

and so on.

We have to be careful, however, since a different choice of basis will give another representation which may look different.

$\rho'(b) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$\sigma(b)$: -1 : eigenvecto
$+1$: eigenvector
σ-

Example 2.4. $\sigma : D_3 \to GL_2(\mathbb{R})$ defined as

$$\boxed{a \overset{\sigma}{\longmapsto} \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}}\ \boxed{b \overset{\sigma}{\longmapsto} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$$

Then, $(\sigma, \mathbb{R}^2) \cong (\rho, V) \cong (\rho', \mathbb{R}^2)$ since the first isomorphism is given by choosing a basis of V.

~~You can see this by figure out the Jordan canonical form of $\sigma(b)$ and $\rho'(b)$; actually, the Jordan decomposition of $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is~~

$$\cancel{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}}$$

~~Thus, the change of basis matrix is one of $\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$.~~

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ -\alpha \end{pmatrix} = \begin{pmatrix} -\alpha \\ \alpha \end{pmatrix} = -\begin{pmatrix} \alpha \\ -\alpha \end{pmatrix}$

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \beta \\ \beta \end{pmatrix} = \begin{pmatrix} \beta \\ \beta \end{pmatrix}$

$\rho' \cong \sigma$.

$\boxed{A = \begin{pmatrix} \alpha & \beta \\ -\alpha & \beta \end{pmatrix}}$

$\exists A \in \mathbb{R}^{2 \times 2}$ s.t.

$\rho'(b) = A^{-1} \sigma(b) A$

$\rho'(a) = A^{-1} \sigma(a) A$

$\boxed{\sqrt{3}\alpha = \beta}$

$A = \begin{pmatrix} 1 & \sqrt{3} \\ -1 & \sqrt{3} \end{pmatrix}$

change