Rep2

Wednesday, September 16, 2020 10:19 AM

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ρ: is irreducible if it has no subrepresentation

$(\rho; V)$, "$\rho(g) = \begin{pmatrix} \rho_1(g) & \\ & \rho_2(g) \end{pmatrix} \in GL(V)$ $\forall g \in G$."

1. Complete Reducibility

Theorem 1.1. *Let V be a representation of finite degree of the finite group G over a field of characteristic not dividing $|G|$. Then every G-invariant subspace $W \subseteq V$ has a G-invariant complement, i.e., $\exists W' \subseteq V$ such that $V = W \oplus W'$ and W' is G-invariant.*

Proof. Given W, we can always find some W'' that is a complement by choosing a basis for W, say $(w_1, \cdots, w_s)$, extending this to a basis of V, say $(w_1, \cdots, w_s, w_{s+1}, \cdots, w_n)$ and then taking $W'' = \text{span}(w_{s+1}, \cdots, w_n)$. The problem is to get G-invariant.

Take W, W'' as above so that $V = W \oplus W''$ and we have the projection $p : V \to V$ sending $w + w''$ to w.

Define $q = \frac{1}{|G|}\sum_{g\in G} \rho(g) \circ p\rho(g^{-1}) \in \text{End}(V)$, meaning

$q = \frac{1}{|G|}\sum_{g\in G} \rho(g)\circ p \circ \rho(g^{-1})$ $$q(v) = \frac{1}{|G|}\sum_{g\in G} \rho(g)(p(\rho(g^{-1})(v)))$$

We claim: $\forall h \in G, \rho(h) \circ q \circ \rho(h^{-1}) = q$. note first that $\{h^{-1}g : g \in G\} = \{g : g \in G\}$, because multiplication by h produces a bijection from LHS to RHS. Thus,

$$\begin{aligned}
\rho(h^{-1}) \circ q \circ \rho(h) &= \rho(h^{-1}) \circ \left(\frac{1}{|G|}\sum_{g\in G} \rho(g) \circ p\rho(g^{-1})\right) \circ \rho(h) \\
&= \frac{1}{|G|}\sum_{g\in G} \rho(h^{-1}) \circ \rho(g) \circ p\rho(g^{-1}) \circ \rho(h) \\
&= \frac{1}{|G|}\sum_{g\in G} \rho(h^{-1}g) \circ p\rho(g^{-1}h) \\
&= \frac{1}{|G|}\sum_{g\in G} \rho(h^{-1}g) \circ p\rho((h^{-1}g)^{-1}) \\
&= \frac{1}{|G|}\sum_{g\in G} \rho(g) \circ p\rho(g^{-1}) \\
&= q.
\end{aligned}$$

linearity

$\{g \in G\} = \{h^{-1}g \in G\}$

□

Let $W' = \ker q$. We claim that

(1) W' is G-invariant.
Let $w' \in W'$, $h \in G$. Then,
$$q(\rho(h)(w')) = (q \circ \rho(h))(w') = \rho(h) \circ \rho(h^{-1}) \circ q \circ \rho(h)(w') = \rho(h) \circ q(w') = \rho(h)(0) = 0.$$
q

Thus, $\rho(h)(w') \in \ker q = W'$.

(2) $\text{im} q \subseteq W$
Take $v \in V$. Then, $\rho(h^{-1})(v) \in V$, thus $p(\rho(h^{-1})(v)) \in W$. Since W is G-invariant, we deduce that $\rho(h)(p(\rho(h^{-1})(v))) \in W$ and so
$$q(v) = \frac{1}{|G|}\sum_{g\in G} \rho(g)(p(\rho(h^{-1})(v))) \in W.$$

(3) $\forall w \in W$, then $q(w) = w$. To see this, $\rho(h^{-1})(w) \in W$. Then, $p(\rho(h^{-1})(w)) = \rho(h^{-1})(w)$, since p is a projection onto W. Thus,
$$\rho(h)(p(\rho(h^{-1})(w))) = \rho(h) \circ \rho(h^{-1})(w) = w.$$

$|G| \cdot w$

Thus,
$$q(w) = \frac{1}{|G|}\sum_{g\in G} \rho(h)(p(\rho(h^{-1})(w))) = \frac{1}{|G|}\sum_{g\in G} w = w.$$

Thus, $q^2(v) = q(q(v)) = q(v)$, since $q(v) \in W$. Thus, $q^2 = q$, hence q is a projection onto W. By (2) and (3), $\text{im} q = W$. So by the lemma of lecture 1,
$$V = \text{im} q \oplus \ker q = W \oplus W'$$
a G-invariant decomposition.

Corollary 1.2 (Maschke's theorem)**.** *Let V be a representation of finite degree of the finite group over a field of characteristic not dividing $|G|$. Then, V is completely reducible.*

Proof. By induction on $\dim V$. If $\dim V = 1$, then V is automatically irreducible and there is nothing to do. (V has no proper subspaces.) Thus, assume $\dim V > 1$. If V is irreducible there is nothing to do. Thus, we can assume that V has a proper G-invariant subspace W. By theorem above, $V = W \oplus W'$ for some G-invariant subspace W'. By induction, both W and W' are completely reducible, i.e.,
$$W = I_1 \oplus \cdots \oplus I_s, W' = I'_1 \oplus \cdots \oplus I'_t$$
where I_j, I'_j are irreducible, and so $V = I_1 \oplus \cdots \oplus I_s \oplus I'_1 \oplus \cdots \oplus I'_t$ is completely reducible. □