$U = \mathbb{C}\{e_1+e_2+e_3\} \to U$

$\oplus\ \mathbb{C}\{e_1-e_2, e_2-e_3\}$

$\hookrightarrow W$

1. Extended example 7.6.

$G = S_3 \qquad V = \mathbb{C}^3.$

2. Let $p: V \longrightarrow U$ projection associated to $U = \boxed{\mathbb{C}(e_1+e_2+e_3)}$

Show $q(v)$ in Maschke's theorem

$$\Rightarrow q\left(\sum_{i=1}^{3} \lambda_i e_i\right) = \boxed{\frac{1}{3}(\lambda_1+\lambda_2+\lambda_3)(e_1+e_2+e_3)}$$

And deduce $\ker q = (e_1-e_2, e_2-e_3)$

pf) Start with basis $\mathbb{C}(e_1+e_2+e_3, e_1, e_2)$

Choice of basis may differ.

$p: V \longrightarrow U$

$\{e_1+e_2+e_3, e_1, e_2\}$

$$\Rightarrow \quad p(e_1) = 0$$
$$p(e_2) = 0$$
$$p(e_3) = p((e_1+e_2+e_3) - e_1 - e_2) = e_1+e_2+e_3$$

Then, $q = \frac{1}{6} \sum_{\sigma \in S_3} \rho(\sigma) \circ p \circ \rho(\sigma^{-1})$

And we knows that

① $\rho((12)) e_1 = e_2$ ② $\rho(13) e_1 = e_2$, ③ $\rho(23) e_3 = e_1$

④ $\rho(123) e_1 = e_2$ ⑤ $\rho(132) e_1 = e_3$

$\Rightarrow$ $p(①) = 0$ $p(②) = e_1 + e_2 + e_3$ $p(③) = 0$

$p(④) = 0$ $p(⑤) = e_1 + e_2 + e_3$

$$\Rightarrow q(e_1) = \frac{1}{6}\left(0 + 0 + (e_1 + e_2 + e_3) + 0 + 0 + (e_1 + e_2 + e_3)\right)$$

$$= \frac{1}{3}(e_1 + e_2 + e_3)$$

By the similar calculation,

$$q(e_2) = q(e_3) = \frac{1}{3}(e_1 + e_2 + e_3)$$

Thus, $q\left(\sum \lambda_i e_i\right) = \frac{1}{3}\boxed{(\lambda_1 + \lambda_2 + \lambda_3)}(e_1 + e_2 + e_3)$

And $\ker q \supseteq \boxed{\sum \lambda_i e_i}$ (affine combination)

if $\boxed{\sum \lambda_i = 0}$ $\Rightarrow$ Basis: $(e_1 - e_2, e_2 - e_3)$

3. Do the samething in (2) but

now let f be projection onto

$\underline{sp(e_1 - e_2, e_2 - e_3)}^{W}$. What is q?

pf) Again, let $\mathbb{C}^3 = sp(e_1 - e_2, e_2 - e_3, e_1)$

then $p(e_1) = 0$

$p(e_2) = p(-(e_1 - e_2) + e_1) = -(e_1 - e_2)$

$= e_2 - e_1$

$p(e_3) = p(-(e_1 - e_2) - (e_2 - e_3) + e_1)$

$= 0 - (e_1 - e_2) - (e_2 - e_3)$

$= e_3 - e_1$

then similar calculation shows that

$$q(e_1) = \frac{1}{3}\left((e_1 - e_2) + (e_1 - e_3)\right)$$

$$q(e_2) = \frac{1}{3}\left((e_2 - e_1) + (e_2 - e_3)\right)$$

$$q(e_3) = \frac{1}{3}\left((e_3 - e_1) + (e_3 - e_2)\right)$$

$$\Rightarrow q\left(\sum \lambda_i e_i\right) = \left(\begin{array}{l} \frac{1}{3}\left((2\lambda_1 - \lambda_2 - \lambda_3)e_1 \right. \\ \quad + (-\lambda_1 + 2\lambda_2 - \lambda_3)e_2 \\ \quad \left. + (-\lambda_1 - \lambda_2 + 2\lambda_3)e_3\right) \end{array}\right)$$

$$\Rightarrow \underline{\ker q} = \left\{ \sum \lambda_i e_i : \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = 0 \right\}$$

which has $\underline{1 \text{ dim}}$ sol space, $\underline{\lambda_1 = \lambda_2 = \lambda_3}$

$$\mathbb{C}\{e_1 + e_2 + e_3\}$$

(4) $\rho : G \longrightarrow GL(2, \mathbb{C})$

degree 2 rep of G.

Show that if $\exists\, g, h \in G$ s.t. $\rho(g)$ and $\rho(h)$ do not commute as a matrix (multiplication) then the representation is irreducible.

Pf)

Suppose ρ is ~~not irreducible~~ reducible.

$\Rightarrow \exists$ basis of $\mathbb{C}^2$ s.t.

$$\rho(g) = \left(\begin{array}{c|c} \boxed{\rho_1(g)} & 0 \\ \hline 0 & \boxed{\rho_2(g)} \end{array}\right)^{2\times 2} \quad \forall g \in G.$$

where $\rho_1(g), \rho_2(g) \in GL_1(\mathbb{C}) = \mathbb{C}^\times$.

$\Rightarrow \rho(g)$ are diagonal, thus commute.

Unitary Representation.

(which gives proof of Maschke's thm ~~on~~ over field $\mathbb{C}$ only.)

V: f.d.v.s with Hermitian form $(\,,\,)$

ex) $(\vec{a}, \vec{b}) := \sum_{i=1}^{n} a_i \bar{b}_i$ $\quad \mathbb{C}^n$

Suppose (V, ρ): rep of G (finite)

$V \simeq \mathbb{C}^n$

Def The Hermitian form is G-inv.

If $(\rho(g)(x), \rho(g)(y)) = (x, y)$ $\forall g \in G$, $x, y \in V$.

and such representation is called

Unitary (b.c. Imp $\in$ Unitary matrices)

(5) $\forall x, y \in V$, define

$$\langle x, y \rangle_1 = \sum_{g \in G} (\rho(g)(x), \rho(g)(y))_0.$$

Show $\langle \, , \, \rangle_1$ is Hermitian form which is G-invariant.

Pf) $\langle x, x \rangle = \sum_{g \in G} (\rho(g)(x), \rho(g)(x))$

$$= \sum_{g \in G} \| \rho(g)(x) \|^2 \geq 0.$$

$\Rightarrow$ it is 0 iff $x = 0$.

Thus the form is positive definite.

check $\langle x, y \rangle = \boxed{\overline{\langle y, x \rangle}}$

$\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$

is tedious .... but held

$\Rightarrow$ Hermitian!

Let $h \in G$.

$\langle \rho(h)(x), \rho(h)(y) \rangle_1$

$= \sum_{g \in G} (\rho(g)\rho(h)(x), \rho(g)\rho(h)y)$

$= \sum_{g \in G} (\rho(gh)(x), \rho(gh)y)$

$= \sum_{g \in G} (\rho(g)(x), \rho(g)y)$

since $\{g \in G\} = \{gh : g \in G\}$

$= \langle x, y \rangle_1 \Rightarrow \langle\ ,\ \rangle$ is G-inv.

(6) Deduce that every complex rep (of finite degree) of G is equiv to a unitary representation.

Pf) Suppose $(V, (\ ,\ )_0)$ exists. Then, by (5) we can construct $\langle\ ,\ \rangle_1$. Now let $(\{\alpha_i\})$ be orthonormal basis w.r.t $\boxed{(\ ,\ )}$ and let $\boxed{\{\beta_i\}}$ be orthonormal basis w.r.t $(\langle\ ,\ \rangle_1)$

$\Rightarrow \exists X \in GL(V)$ s.t.

$$\boxed{(x, y)_0 = \langle Xx, Xy \rangle}$$

(X : basis change matrix.)

Thus, given representation

$\rho : G \to GL(V)$, define

$\boxed{\sigma : G \to GL(V)}$ s.t.

$$\sigma(g) = X^{-1} \rho(g) X .$$

$$\Rightarrow \big(\sigma(g)(x), \sigma(g)(y) \big)$$

$$= \big(X^{-1} \rho(g) X x, X^{-1} \rho(g) X y \big)_0$$

$$= \langle \rho(g) X x, \rho(g) X y \rangle_1$$

$$= \langle X x, X y \rangle$$

$$= (x, y) \quad \# \quad \text{over } (\; , \;)_0$$

$\Rightarrow \boxed{\sigma}$ is unitary rep. equiv to ρ.

(7) U : subrepresentation of $V \cong \mathbb{C}^n$

$\Rightarrow$ $U^\perp = \{ v \in V : \langle u, v \rangle = 0 \ \forall u \in U \}$

is G-invariant subspace of V

which is complementary to U.

pf) Let $v \in U^\perp$, then $\forall u \in U$

$\forall g \in G$

$$\langle u, \rho(g) v \rangle \underset{G\text{-inv.}}{=} \langle \rho(g)^{-1} u, v \rangle = 0$$

$\in U$

Since $\rho(g)^{-1} u \in U$, $v \in U^\perp$.

$\Rightarrow$ $\rho(g) v \in U^\perp$ $\Rightarrow$ So $U^\perp$ is G-inv.

Now $\boxed{V = U \oplus U^\perp}$ (since ① $U \cap U^\perp = \{0\}$

$\langle \ , \ \rangle$: non-degenerate pairing.

$\langle v, u \rangle = 0 \Leftarrow \langle v \in U, u \rangle = 0 \ \forall u \in U$.

(8). Deduce Maschke's Thm

Pf) Let V be a rep of G.

WLOG, assume V is unitary, by taking (5)–(7).

Let U be a subrep. Then by (7) U has a G-inv complement.