6. Isotypic Decomposition

Assume G is finite, $|G|^{-1} \in F$

Maschke's Theorem (from Lecture 4) tells us that a finite deg. rep. V can be decomposed into a direct sum of irred. reps.: $V = V_1 \oplus \cdots \oplus V_k$. How unique is this? What if $V = V_1' \oplus \cdots \oplus V_j'$. Is there a relationship between these decompositions? The answer is yes! And the isotypic decomposition is what we're looking for.

Lemma 6.1. *Let $\phi : V \to W$ be a G -homomorphism. Then V decomposes into a direct sum of G -invariant subspaces $V = U \oplus \ker\phi$ where $U \cong \mathrm{im}\phi \subset W$*

Proof. Recall, ker ϕ is a subrep. of V. By a theorem from Lecture 4 (Theorem 4.1), we can find a G -invariant complement U to ker $\phi : V = U \oplus \ker\phi$. We just need to show that $U \cong \mathrm{im}\phi$. Let $\phi|_U : U \to W$ be the restriction of ϕ to U. since ϕ is a G -homomorphism, so is $\phi|_U$. Also, if $u \in U$, then $u \in \ker \phi|_U \iff u \in \ker \phi \cap U = \{0\}$. This is $\{0\}$ by our definition of U as a complement of ker ϕ. Thus, $u = 0$, and $\phi|_U$ is injective.
Next, we'll show that $\phi|_U$ has image im ϕ. We will show $\mathrm{im}\phi|_U = \mathrm{im}\phi$. First, note that $\mathrm{im}\phi|_U \subset \mathrm{im}\phi$ For the reverse inclusion, suppose $w \in \mathrm{im}\phi$. Then, for some $v \in V, w = \phi(v)$. But, $v = u+k$. So, $w = \phi(u+k) = \phi(u)+\phi(k) = \phi(u) = \phi|_U(u)$. Thus, $w \in \mathrm{im}\phi|_U$. So, $\phi|_U$ is an injective G -homomorphism whose image is im ϕ. Thus, $U \cong \mathrm{im}\phi$. □

Definition 6.2. Let V be a representation of G of finite degree. By Maschke's theorem, $V = V_1 \oplus \cdots \oplus V_k$ where each V_i is an irreducible representation of G. After re-odering we may assume that the decomposition is arranged

$$\begin{aligned} V = & \underbrace{V_1 \oplus \cdots \oplus V_{n_1}}_{\text{all } G\text{-isomorphic to } I_1} \oplus \underbrace{V_{n_1+1} \oplus \cdots \oplus V_{n_2}}_{\text{all } G\text{-isomorphic to } I_2} \oplus \cdots \oplus \underbrace{V_{n_{m-1}+1} \oplus \cdots \oplus V_{n_m}}_{\text{all } G\text{-isomorphic to } I_m} \\ & = V^{I_1} \oplus V^{I_2} \oplus \cdots \oplus V^{I_n}. \end{aligned}$$

with $k = n_m$, $I_1, \cdots, I_m$ are irreducible G-representation and $I_s \ncong I_t$ for all $s \neq t$. So the V_i's that are all isomorphic to one another are collected together and each collection consists only of one isomorphic class. Such a decomposition is called an *isotypic decomposition of* V and each V^{I_j} an *isotypic component* (associated to I_j).

Theorem 6.3. *Take $V = V^{I_1} \oplus \cdots \oplus V^{I_m}$ as above.*

(1) Suppose $U \subseteq V$ is an irreducible subrepresentation. Then $U \cong V_i$ for some i and U is a subspace of the isotypic V^{I_j} that contains V_i.

(2) The isotypic decomposition is unique, i.e., $V = W_1 \oplus \cdots \oplus W_e$ is another decomposition of V into irreducible subrepresentation then the resulting isotypic decomposition equals the one above $V = V^{I_1} \oplus \cdots \oplus V^{I_m}$.

(3) The number of times an irreducible I appears in a decomposition (up to isomorphism) is independent of the decomposition.

Proof. (1) Define a mapping $\phi_j : U \to V_j$ as follow.
Notes that $u \in U \subseteq V$ so write $u = v_1 + \cdots + v_k$ with $v_i \in V_i$, $(1 \leq i \leq k)$. Then $\phi_j(u) = v_j$. This is a G-homomorphism. (Check!) Suppose $u \neq 0$. Then $\exists j$ such that $v_j \neq 0$ and hence $\phi_j \neq 0$. Since U and V_j are both irreducible, and ϕ_j is a non-zero G-homomorphism between them, ϕ_J is a G-isomorphism by Theorem 5.7. So $U \cong V_j$.
(Recall Theorem 5.7: if $\varphi : V \to W$ is a G-homomorphism between two irreducible representation of G, then φ is either zero map or isomorphism.)

It follows that for any $u = v_1 + \cdots + v_k$, then we have $v_j \neq 0$ if $u \neq 0$. Suppose that $v_l \neq 0$ for some $u \in U$. Then by the same argument, ϕ_l is an isomorphism and hence $U \cong V_l$. Thus, $V_l \cong V_j$, which implies that V_l and V_j are in the same isotypic component, thus U is inside of an isotypic component containing V_j.
(2) Let $V = W_1 \oplus \cdots \oplus W_l$ be another decomposition into irreducible subrepresentations. If $W_{i_1}, \cdots, W_{i_t}$ are the irreducible representation in this decomposition that are isomorphic to I_j, then by (1), each $W_{i_1}, \cdots, W_{i_t} \subseteq V^{I_j}$, hence $W_{i_1} \oplus \cdots \oplus W_{i_t} \subseteq V^{I_j}$. Thus the isotypic components w.r.t. the 2nd decomposition are subspaces of the isotypic components w.r.t. 1st decomposition. The same argument swapping the roles of the 2nd decomposition shows that isotypic components w.r.t 1st decomposition lies inside of isotypic components of the 2nd decomoposition. Thus we have equality and uniqueness.
(3) This is clear;

$$\dim V^{I_j} = (\#\text{irreducible representation} \cong I_j \text{ in the decomposition of } V) \cdot \dim I_j$$

and by (2), the LHS is independent of the decomposition. □

Example 6.4. The above analysis applied to $V = F[G]$ is interesting. Let I be any irreducible representation of G and pick $z \in I$ ($z \neq 0$). Define $\phi : F[G] \to I$ by

$$\phi(\sum_{g \in G} \lambda_g v_g) = \sum_{g \in G} \lambda_g \rho_I(g)(z)$$

for some $\lambda_g \in F$. Then we claim that ϕ is a G-homomorphism.

(1) Linearity: $\phi(\sum_{g \in G} \lambda_g v_g + \sum_{g \in G} \mu_g v_g) = \cdots = \sum_{g \in G} (\lambda_g + \mu_g) \rho_I(g)(z) = \cdots = \phi(\sum_{g \in G} \lambda_g v_g) + \phi(\sum_{g \in G} \mu_g v_g)$
(2) G-intertwining: for $h \in G$,

$$\phi(\rho_{F[G]}(h) \cdot \sum_{g \in G} \lambda_g v_g) = \phi(\sum_{g \in G} \lambda_g v_{hg}) = \sum_{g \in G} \lambda_g \rho_I(hg)(z) = \sum_{g \in G} \lambda_g \rho_I(h)(\rho_I(g)(z))$$
$$= \rho_I(h) \left(\sum_{g \in G} \lambda_g (\rho_I(g)(z)) \right) = \rho_I(h) \phi(\sum_{g \in G} \lambda_g v_g)$$

By Lemma 6.1, $F[G] = \ker \phi \oplus U$ where $U \cong \operatorname{im} \phi$. But $\phi \neq 0$ since $0 \neq z = \phi(v_e) \in \operatorname{im} \phi$. Since I is irreducible and $\operatorname{im} \phi$ is a nonzero subrepresentation of I, we get $\operatorname{im} \phi = I$. Thus there is a subrepresentation of $F[G]$ isomorphic to I.

Corollary 6.5. *There are a finite number of irreducible G-representation (up to G-isomorphism).*

Proof. Let I be an irreducible G-representation. By the above, there is a subgroup of $F[G]$ isomorphic to I and thus $F[G]$ has a non-zeo isotypic component associated to I. By Theorem 6.3, there are only a finite number of isotypic component and so there are only a finitely many Is (up to G-isomorphism). □