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8. Class functions

Definition 8.1. A class function on G is a function $f : G \to \mathbb{C}$ such that $f(g) = f(xgx^{-1})$ for all $x, g \in G$. In other words, f is a complex-valued function on G that is constant on conjugacy classes.

Example 8.2.

(1) Any character χ_V is a class function.
(2) Let $Cl(g)$ be the conjugacy class of g in G. Then, the following is a class function

$$ch_{Cl(g)}(x) = \begin{cases} 1 & \text{if } x \in Cl(g) \\ 0 & \text{if } x \notin Cl(g) \end{cases}$$

(3) The following is a class function if and only if $g \in Z(G)$ the center of the group G :

$$ch_g(x) = \begin{cases} 1 & x = g \\ 0 & x \neq g \end{cases}$$

Definition 8.3. $\mathscr{C} := \{f : G \to \mathbb{C} : f \text{ is a class function}\}$ is the *space of class functions* of G. Given $f_1, f_2 \in \mathscr{C}$, $\langle f_1, f_2\rangle := \frac{1}{|G|}\sum_{g\in G}\overline{f_1(g)}f_2(g)$.

$(f_1(g_1), f_1(g_2), \cdots f_1(g_n)) \cdot (f_2(g_1), f_2(g_2)$

Proposition 8.4. .

(1) $\mathscr{C}$ is a $\mathbb{C}$-vector space.
(2) Let $C_1, \cdots, C_m$ be the complete list of mutually distinct conjugacy classes of G. Then, the function $ch_{C_1}, \cdots, ch_{C_m}$ form a basis of $\mathscr{C}$.
(3) $\dim \mathscr{C} = \#$ conjugacy classes in G.

Proof. (1) We know already that $\mathbb{C}[G] = \{f : G \to \mathbb{C}\}$ is a $\mathbb{C}$-vector space, so we want to prove that $\mathscr{C}$ is a subspace of $\mathbb{C}[G]$. Let $f_1, f_2 \in \mathscr{C}$ and $\lambda, \mu \in \mathbb{C}$. Then, by definition,

$$(\lambda f_1 + \mu f_2)(g) = (\lambda f_1)(g) + (\mu f_2)(g) = \lambda(f_1(g)) + \mu(f_2(g)) = \lambda(f_1(x^{-1}gx)) + \mu(f_2(x^{-1}gx)) = (\lambda f_1 + \mu f_2)(x^{-1}gx)$$

thus $(\lambda f_1 + \mu f_2) \in \mathscr{C}$ as desired.
(2) Let $f \in \mathscr{C}$ and $g_i \in C_i$ for $1 \le i \le m$. Then we claim that

$$\boxed{f = \sum_{i=1}^{m} f(g_i) ch_{C_i}.}$$

To see this, let $x \in G$. Then, $x \in C_j$ for some unique j, $1 \le j \le m$. So,

$$\left(\sum_{i=1}^{m} f(g_i) ch_{C_i}\right)(x) = \sum_{i=1}^{m} f(g_i) ch_{C_i}(x) = f(g_1)\cdot 0 + f(g_2)\cdot 0 + \cdots + f(g_j)\cdot 1 + \cdots + f(g_m)\cdot 0 = f(g_j) = f(x)$$

since $x, g_j \in C_j$ and $f \in \mathscr{C}$.
Thus, $ch_{C_1}, \cdots, ch_{C_m}$ are a spanning set for $\mathscr{C}$. To prove the linear independence, suppose that

$$\sum_{i=1}^{m} \lambda_i ch_{C_i} = 0$$

where $\lambda_i \in \mathbb{C}$. Evaluating at g_j for $1 \le j \le m$ produces

$$0 = \sum_{i=1}^{m} \lambda_i ch_{C_i}(g_j) = \lambda_j$$

as desired.
(3) Consequence of (2). □

Theorem 8.5. *Let $I_1, \cdots I_n$ be a complete list of non-isomorphic irreducible representations of G. Then, $\chi_{I_1}, \cdots, \chi_{I_n}$ form a basis of $\mathscr{C}$.*

To see this, we need a prepartory lemma

$\rho_V : \mathscr{C} \to GL(V)$ $\rho_V(f)$:

Lemma 8.6. *Let $f \in \mathscr{C}$ and (ρ, V) is a representation of G. Define a new linear function $\rho_V : V \to V$ such that*

$$\rho_V(f) = \sum_{g\in G} \overline{f(g)}\rho(g).$$

Then, $\rho_V(f)$ is G-intertwining and $\rho_V(g) = \frac{|G|}{\dim V}\langle f, \chi_V\rangle \mathrm{Id}_V$

$\rho(h^{-1})\left(\sum_{g\in G} \overline{f(g)}\rho(g)\right)\rho(h)$

Proof.

$$\begin{aligned} \rho(h^{-1})\rho_V(g)\rho(h) &= \sum_{g\in G}\rho(h^{-1})\overline{f(g)}\rho(g)\rho(h) \\ &= \sum_{g\in G}\overline{f(g)}\boxed{\rho(h^{-1})\rho(g)\rho(h)} \\ &= \sum_{g\in G}\overline{f(g)}\rho(h^{-1}gh) \\ &= \sum_{g\in G}\overline{f(h^{-1}gh)}\rho(h^{-1}gh) \\ &= \sum_{g\in G}\overline{f(g)}\rho(g) = \boxed{\rho_V(f)} \end{aligned}$$

$f \in \mathscr{C}$

$f(g) = f(h^{-1}gh)$

$\rho_V(f) = \frac{\mathrm{Tr}(\rho_V(f))}{\dim V}\mathrm{Id}_V.$ $\mathrm{Tr}(\rho_V(f)) = |G| \langle f, \chi_V\rangle$

Hence, $\rho_V(f)$ is G-intertwining.
If V is irreducible, then by Schur's lemma, $\rho_V(g) = \boxed{\frac{\mathrm{Tr}(\rho_V(g))}{\dim V}\mathrm{Id}_V}$. Thus, it suffices to show that $\mathrm{Tr}(\rho_V(g)) = |G|\langle f, \chi_V\rangle$. To see this, observe that

$$\mathrm{Tr}(\rho_V(g)) = \mathrm{Tr}\Big(\sum_{g\in G}\overline{f(g)}\rho(g)\Big) = \sum_{g\in G}\overline{f(g)}\,\mathrm{Tr}(\rho(g)) = \sum_{g\in G}\overline{f(g)}\chi_V(g) = |G|\left(\frac{1}{|G|}\sum_{g\in G}\overline{f(g)}\chi_V(g)\right) = |G|\langle f, \chi_V\rangle$$

□

$\{\chi_{I_1}, \cdots \chi_{I_n}\} \to \{\chi_{I_1}, \cdots \chi_{I_n}, f_1, f_2, f_3\}$

Now, we can see the proof of theorem 8.5.

Proof of theorem 8.5. We have already shown that $\chi_{I_1}, \cdots, \chi_{I_n}$ are orthogonal elements of $\mathscr{C}$ w.r.t. $\langle,\rangle$. Any such set can be extended to an orthogonal basis of $\mathscr{C}$ (using Gram-Schmidt). So if we show that there are no further elements of $\mathscr{C}$ that are orthogonal to $\chi_{I_1}, \cdots, \chi_{I_n}$ then this list must inded form a basis.
Let $f \in \mathscr{C}$ such that $\boxed{\langle f, \chi_{I_j}\rangle = 0}$ for all j, $1 \le j \le n$. We will show that $f = 0$. To see this, observe that if we

decompose $\mathbb{C}[G] = U_1 \oplus \cdots \oplus U_t$, then

$$\begin{aligned} \boxed{\rho_{\mathbb{C}[G]}(f)} &= \sum_{g\in G}\overline{f(g)}\rho_{\mathbb{C}[G]}(g) = \sum_{g\in G}\overline{f(g)}\begin{pmatrix} \rho_{U_1}(g) & 0 & \cdots & 0 \\ 0 & \rho_{U_2}(g) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \rho_{U_t}(g) \end{pmatrix} \\ &= \begin{pmatrix} \sum_{g\in G}\overline{f(g)}\rho_{U_1}(g) & 0 & \cdots & 0 \\ 0 & \sum_{g\in G}\overline{f(g)}\rho_{U_2}(g) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \sum_{g\in G}\overline{f(g)}\rho_{U_t}(g) \end{pmatrix} \\ &= \begin{pmatrix} \frac{|G|}{\dim U_1}\langle f, \chi_{U_1}\rangle \mathrm{Id}_{U_1} & 0 & \cdots & 0 \\ 0 & \frac{|G|}{\dim U_2}\langle f, \chi_{U_2}\rangle \mathrm{Id}_{U_2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \frac{|G|}{\dim U_t}\langle f, \chi_{U_t}\rangle \mathrm{Id}_{U_t} \end{pmatrix} \\ &= 0 \text{ by hypothesis on } f. \end{aligned}$$

Then, from

$$\rho_{\mathbb{C}[G]}(f)(v_e) = \sum_{g\in G}\overline{f}(g)\rho_{\mathbb{C}[G]}(g)(v_e) = \sum_{g\in G}\overline{f(g)}v_g = 0$$

where 0 is from the above calculation of $\rho_{\mathbb{C}[G]}(f)$, we can conclude that $\overline{f}(g) = 0$ for all g, which implies $f = 0$. □

Corollary 8.7. *The number of complex irreducible representation of G is equal to the number of conjugacy classes of G.*

Definition 8.8. Let $I_1, \cdots, I_m$ be the $\mathbb{C}$-irreducible representations of G and $Cl_1, \cdots, Cl_m$ be the conjugacy classes of G. Then, the *character table* of G is the $m \times m$ table where (i,j)th entry is $\chi_{I_i}(g_j)$ for some $g_j \in Cl_j$.

Example 8.9. .

(1) $G = C_n = \langle x\rangle$ is abelian. Then, $Cl_j = \{x^j\}$ for $j \in \{0, 1, \cdots n-1\}$. Then, $\omega = e^{2\pi\frac{\sqrt{-1}}{n}}$, I_j is given by $\rho(x) = \omega^j = e^{2\pi\frac{\sqrt{-1}j}{n}}$. Then,

ω nth of unity

$\omega^{2(n-1)} = \omega^n \cdot \omega^{n-2}$

	e	x	x^2	$\cdots$	x^{n-1}
I_0	1	1	1	$\cdots$	1
I_1	1	ω	ω^2	$\cdots$	ω^{n-1}
I_2	1	ω^2	ω^4	$\cdots$	ω^{n-2}
$\cdots$	$\cdots$				
I_{n-1}	1	ω^{n-1}	ω^{n-2}	$\cdots$	ω

$\rho_i(x^j) = \omega^{ij}$

from the fact that $\omega^{2(n-1)} = \omega^n\omega^{n-2} = \omega^{n-2}$ and $\omega^{(n-1)(n-1)} = \omega^{n^2+2n+1} = \omega$.

(2) If $G = S_3$, then conjugacy classes are determined by cycle length, thus it has 3 conjugacy classes. Here, I_1 is the trivial representation, I_2 is the signed representation by sending $\pi \mapsto \mathrm{sign}(\pi)$, and I_3 is $\mathbb{C}^2$ subrepresentation of $\mathbb{C}^3$ spanned by $e_1 - e_2, e_2 - e_3$ where

$$\begin{cases} e_1 - e_2 & \mapsto e_2 - e_3 \\ e_2 - e_3 & \mapsto \boxed{e_3 - e_1} = -(e_2 - e_3) - (e_1 - e_2) \end{cases}$$

$\{e\}$
$\{(12), (13), (23)\}$
$\{(123), (132)\}$

$e_1 = (1,0,0)$
$e_2 = (0,1,0)$
$e_3 = (0,0,1)$

Thus, the character table is

	e	(12)	(123)
I_1	1	1	1
I_2	1	-1	1
I_3	2	0	-1

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\rho((12)) = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$ $\rho((123)) =$

(3) $G = D_4 = \langle a, b : a^4 = b^2 = 1, bab^{-1} = a^{-1}\rangle$. We knows that the conjugacy classes are

$$\{e\}, \{a, a^3\}, \{bba^2\}, \{ab, a^3b\}, \{a^2\}.$$

Hence we have $\boxed{5 \text{ irreducible representations.}}$ Then, 1-dim representations should be $a \mapsto \alpha \in \mathbb{C}, b \mapsto \beta \in \mathbb{C}$ satisfying $\alpha^4 = \beta^2 = 1, \alpha = \alpha^{-1}$. Thus,

$$\beta^2 = \alpha^2 = 1 \implies \boxed{\beta = \pm 1, \alpha = \pm 1.}$$

Thus, we get for irreducible 1-dimensional representations. Moreover, we have a 2 dimensional natural representation of G such that $\rho(a) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\rho(b) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, which gives the character below.

$$\chi_V(e) = 2, \chi_V(a) = \chi_V(b) = \chi_V(ab) = 0, \chi_V(a^2) = -2.$$

$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

from

$$\langle \chi_V, \chi_V\rangle = \frac{1}{|G|}\sum_{g\in G}\overline{\chi_V(g)}\chi_V(g) = \frac{1}{8}(2^2 + 0 + 0 + 0 + 2^2), = 1$$

we can fill out the table as below

	e	a	b	ab	a
χ_1	1	1	1	1	1
χ_2	1	-1	1	-1	1
χ_3	1	1	-1	-1	1
χ_4	1	-1	-1	1	1
χ_V	2	0	0	0	-2

9. Lifting characters

Suppose $N \trianglelefteq G$. We would like to compute the representations of G/N with the representation of G. Suppose we have $\tilde{\rho} : G/N \to GL(V)$ representations of G/N. Then, we get for free a representation of G

$$\rho : G \to G/N \xrightarrow{\tilde{\rho}} GL(V) \text{ by } g \mapsto gN \mapsto \tilde{\rho}(gN).$$

$g, g' \in gN$

For characters, $\chi_\rho(g) = \mathrm{Tr}(\rho(g)) = \mathrm{Tr}(\tilde{\rho}(gN)) = \chi_{\tilde{\rho}}(gN)$.

Definition 9.1. If $N \trianglelefteq G$ and $\tilde{\chi}$ is a character of G/N, then the character χ which is given by $\chi(G) = \chi(\tilde{g}N)$ is called the *lift of $\tilde{\chi}$ to G.*

Notes that $\chi(e) = \tilde{\chi}(eN)$, hence $\tilde{\chi}$ is irreducible iff χ is irreducible.

$\chi(g) = \chi(g') = \tilde{\chi}(gN)$

Lemma 9.2. *Let $\rho : G \to GL(V)$ representation. Then, $g \in \ker(\rho) \iff \chi_\rho(g) = \chi_\rho(e)$.*

$\rho(g) = \mathrm{Id}_V \Rightarrow$

Proposition 9.3. Let χ be a character of G. Then,

(1) $\ker(\chi) := \{g \in G : \chi(g) = \chi(e)\} \trianglelefteq G$
(2) χ can be lifted from $G/\ker(\chi)$.

$\mathrm{Tr}(\rho(g)) = \mathrm{Tr}(\mathrm{Id}_V) = \mathrm{Tr}(\rho(e))$

Proof. (1): Let $\chi = \chi_{V,\rho}$. By the above lemma, $\ker(\chi) = \ker(\rho) \trianglelefteq G$.
(2) Let $K = \ker(\chi)$. We can define $\tilde{\rho} : G/K \to GL(V)$ a representaion by setting $\tilde{\rho}(gK) = \rho(g)$. This is well-defined since $gK = g'K$ implies $g^{-1}g' \in K$, thus $\rho(g^{-1}g') = \mathrm{Id}_V$, hence $\rho(g) = \rho(g')$. By this construction, χ is the lift of $\chi_{\tilde{\rho}}$. □

$\rho(g') = \rho(g)$

$gK = g'K$ $G/K \to GL(V)$ $\tilde{\rho}(gK) = \rho(g)$

$g^{-1}g' \in K$ $\rho(g^{-1}g') = \mathrm{Id}_V \Rightarrow \rho(g^{-1})\rho(g') = \mathrm{Id}_V$

Corollary 9.4. *G is not simple $\iff \exists g \in G \setminus \{e\}$ and a nontrivial irreducible character χ such that $\chi(g) = \chi(e)$.*

Proof. Suppose the righthandside. Then, set $K_\chi := \ker\chi \trianglelefteq G$ by the theoem. Then, $K_\chi \neq G$ because χ is nontrivial and $\{e\} \neq K_\chi$ because $g \in K_\rho$. Thus, G is not simple.
Suppose the G is not simple. Take $N \trianglelefteq G$, $\{e\} \neq N \neq G$. Then, let $\tilde{\chi}$ be a nontrivial irreducible characer of G/N. Lift $\tilde{\chi}$ to a character χ of G. Then, χ is irreducible and $\chi(g) = \tilde{\chi}(gN) = \tilde{\chi}(eN) = \chi(e)$ for all $g \in N$. □