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Wednesday, November 25, 2020 10:14 AM

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11. Induced representation

Review for last time.

$G \geq H$.

Definition 11.1. Given (V, ρ) of G, we can consider the reprsentation of H given by restriction $(\mathrm{res}_H^G V) : H \hookrightarrow G \to GL(V)$

Lemma 11.2. *Let χ_V be the character of V. Then, $\chi_{\mathrm{res}_H^G V} = \chi_V|_H$.*

Definition 11.3. Definition: Let (W, ρ_W) be a representation of H. The induced representation $\mathrm{Ind}_H^G W$ is the space

$$\{f : G \to W \mid f(gh) = \rho_W(h)^{-1} f(g) \quad \forall h \in H, \forall g \in G\}$$

Lemma 11.4. *$\mathrm{Ind}_H^G W$ is a representation of G*

$gH = g'H \Rightarrow f(g) = f(g')$

$f_{w,g}(gh) = f_{w_{g'}}(g'h)$

$gh \in gH = g'H \ni g'h$

Example 11.5.

- $H = G$; $\mathrm{Ind}_H^G W \cong W$. We just send f to $f(e_G)$
- $H = \{e\}$; $\mathrm{Ind}_H^G \mathbb{C} = \mathbb{C}[G]$, the regular representation.
- $\mathrm{Ind}_{A_4}^{A_5} \mathbb{C} \cong I_1 \oplus I_2$(see Exercise 3 in Lec 13/14 wkst)

$gH = g'H \Rightarrow$ " $W_g = W_{g'}$

Start new! $W_r \quad r \in G/H$.

Let

$$W_g := \{f_{w,g} : G \to W : w \in W, g \in G\} \text{ where } \begin{cases} f_{w,g}(x) := 0 & \text{if } x \notin gH \\ f_{w,g}(gh) := \rho_W(h^{-1})w & \forall h \in H \end{cases}$$

Obviously, $\mathcal{O}_g : W_g \to W$ by $\mathcal{O}_g(f_{w,g}) := w$ is a vector space isomorphism. Moreover, W_g can be denoted as W_{gH} since it is invariant under the coset. (In other words, if $gH = g'H$, then $W_g = W_{g'}$; just permuting W_g.)

Theorem 11.6.

(1) $\mathrm{Ind}_H^G(W \oplus W') = \mathrm{Ind}_H^G W \oplus \mathrm{Ind}_H^G W'$

(2) Let R be a set of coset representation of G/H. Then, as a vector space, $\mathrm{Ind}_H^G W = \oplus_{r\in R} W_r$. In particular, $\dim(\mathrm{Ind}_H^G W) = [G : H] \dim W$.

(3) Let $H \leq K \leq G$. Then $\mathrm{Ind}_H^G W \cong \mathrm{Ind}_K^G (\mathrm{Ind}_H^K W)$.

Proof. (1): For any $f : G \to W \oplus W'$ we may write $f(g) = (f_1(g), f_2(g))$ with $f_1 : G \to W$, $f_2 : G \to W'$, conversely, for any such f_1 and f_2, we may compose them into a function $(f_1, f_2) : G \to W \oplus W'$. This gives an isomorphism between them.

(2): Since $gH = g'H$ implies $W_g = W_{g'}$, we may write W_r for all W_g with $r \in R$. Since $W_r \subseteq \mathrm{Ind}_H^G W$ is clear, it suffices to show that $\sum_{r\in R} W_r$ is a direct sum, and $\mathrm{Ind}_H^G W$ are contained in the direct sum.

To see $\sum_{r\in R} W_r = \oplus_{r\in R} W_r$, let $f \in W_r \cap \sum_{s\neq r} W_s$. Then, $f = f_{w,r} = \sum_{s\neq r, w\in W} \lambda_{w,s} f_{w,s}$ for some $\lambda_{w,s} \in \mathbb{C}$. If $gH = r$, then $f_{w,s}(g) = 0$ for all $s \neq r$, thus

$$f(g) = \sum_{w\in W} \lambda_{w,s} f_{w,s}(g) = 0.$$

However, if $gH \neq r$, then $f(g) = f_{w,r}(g) = 0$, thus $f = 0$. This shows that $W_r \cap \sum_{s\neq r} W_s = 0$, hence the sum is actually the direct sum.

Now, to see $\oplus_{r\in R} W_r = \mathrm{Ind}_H^G W$, let $f \in \mathrm{Ind}_H^G W$. Let $w_r = f(r)$ for all $r \in R$. We claim that $f = \sum_{r\in R} f_{w_r,r} \in \oplus_{r\in R} W_r$. To see this, let $g \in G$. Thus, there exists some $s \in R, h \in H$ such that $g = sh$. Then,

$$f(g) = f(sh) = \rho_W(h)^{-1} f(s) = \rho_W(h^{-1}) w_s = f_{w_s,s}(sh) = \sum_{r\in R} f_{w_r,r}(sh) = \left(\sum_{r\in R} f_{w_r,r}\right)(g).$$

$= f_{w_s,s}(sh) + \sum_{r\neq s} f_{w_r,r}(sh)$

Thus, $\oplus_{r\in R} W_r = \mathrm{Ind}_H^G W$. Since $\mathcal{O}_g : W_g \to W$ is a vector space isomorphism, we deduce that

$$\dim \mathrm{Ind}_H^G W = \sum_{r\in R} \dim W_r = |R| \dim W = [G : H] \dim W.$$

(3): Exercise. □

We want to consider how G acts on each W_r appearing in (b). We claim that

$$\rho_{\mathrm{Ind}_H^G W}(g)|_{W_r} : W_r \to W_{r'}$$

$f_{w,r}$ for some w

where r' is defined by $r'H = grH$. To check this, let $f_r \in W_r$. Then, for $s \in R$, $h \in H$,

$$\rho_{\mathrm{Ind}_H^G W}(g)(f_r)(sh) = f_r(g^{-1}sh) = \begin{cases} 0 & \text{if } g^{-1}sH \neq rH \\ f_r(rH) & \text{if } g^{-1}sH = rH \end{cases}$$

$\in W_r$?

Notation: Suppose we have a class function χ on H. Write $\dot\chi$ for the function $G \to \mathbb{C}$ (not necessarily a class function)

$$\dot\chi = \begin{cases} \chi(g) & \text{if } g \in H \\ 0 & \text{if } g \notin H \end{cases}.$$

$\dot\chi = \chi|_H$.

Definition 11.7. Let $\chi \in \mathcal{C}_H$. Then, define $\chi\uparrow_H^G : G \to \mathbb{C}$ by

$$\chi\uparrow_H^G(g) = \frac{1}{|H|}\sum_{x\in G} \dot\chi(x^{-1}gx) = \sum_{r\in R} \dot\chi(r^{-1}gr)$$

Notes that the last equality holds since $x = rh$ implies $\dot\chi(h^{-1}r^{-1}grh) = \dot\chi(r^{-1}gr)$.

Theorem 11.8. *(1) $\chi\uparrow_H^G \in \mathcal{C}_H$.*

(2) $\chi_{\mathrm{Ind}_H^G W} = \chi_W\uparrow_H^G$.

$\sum_{x\in G} \dot\chi((yx)^{-1} g (yx)) = \sum \dot\chi(z^{-1} g z)$

$z = yx \in G$.

Proof. (1) Let $g, y \in G$. Then,

$$\chi\uparrow_H^G(y^{-1}gy) = \frac{1}{|H|}\sum_{x\in G}\dot\chi(x^{-1}y^{-1}gyx) = \frac{1}{|H|}\sum_{x\in G}\dot\chi((yx)^{-1}g(yx)) = \frac{1}{|H|}\sum_{x\in G}\dot\chi(x^{-1}gx) = \chi\uparrow_H^G(g).$$

(2): We want to calculate $\chi_{\mathrm{Ind}_H^G W}(g)$. Writing $\mathrm{Ind}_H^G W = \oplus_{r\in R} W_r$, we see that we can represent the matrix of $\rho_{\mathrm{Ind}_H^G W}(g)$ as an $|R| \times |R|$ block matrix. Now, the trace of this matrix can only pick up entries for r such that W_r goes to W_r under $\rho_{\mathrm{Ind}_H^G W}(g)$. Thus,

$$\chi_{\mathrm{Ind}_H^G W}(g) = \sum_{r\in R, r^{-1}gr\in H} \mathrm{Tr}_{W_r}(\rho_{\mathrm{Ind}_H^G W}(g)|_{W_r}). \quad (+)$$

W_r W_r

Tr_{W_r}(

We have $W \xrightarrow{\mathcal{O}_r^{-1}} W_r \xrightarrow{\rho_{\mathrm{Ind}_H^G W}(g)} W_r \xrightarrow{\mathcal{O}_r} W$, which gives

$$\mathrm{Tr}_{W_r}(\rho_{\mathrm{Ind}_H^G W}(g)) = \mathrm{Tr}_W(\mathcal{O}_r \circ \rho_{\mathrm{Ind}_H^G W}(g) \circ \mathcal{O}_r^{-1}) \quad (++)$$

Now,

$$\rho_{\mathrm{Ind}_H^G W}(g)(f_{w,r})(rh) = f_{w,r}(g^{-1}rh) = f_{w,r}(rr^{-1}g^{-1}rh) = f_{w,r}(rk^{-1}h) = \rho_W(k^{-1}h)^{-1}w = \rho_W(h)^{-1}\rho_W(k)w = f_{\rho_W(k)w,r}(rh).$$

where $k = r^{-1}gr \in H$. Thus,

$k = r^{-1}gr$. $= f_{\rho_W(k)w,r}$ $\rho_W(h)^{-1}\rho_W(k)w$.

$$\mathcal{O}_r\rho_{\mathrm{Ind}_H^G W}(g)\mathcal{O}_r^{-1}(w) = \mathcal{O}_r\rho_{\mathrm{Ind}_H^G W}(g) f_{w,r} = \mathcal{O}_r f_{\rho_W(k)w,r} = \rho_W(k)w.$$

Thus,

$$\mathrm{Tr}_W(\mathcal{O}_r\rho_{\mathrm{Ind}_H^G(g)}\mathcal{O}_r^{-1}) = \mathrm{Tr}(\rho_W(k)) = \chi_W(k) = \chi_W(r^{-1}gr).$$

Hence, (+) and (++) implies that

$$\chi_{\mathrm{Ind}_H^G W}(g) = \sum_{r\in R, r^{-1}gr\in H} \chi_W(r^{-1}gr) = \sum_{r\in R}\dot\chi_W(r^{-1}gr) = \chi_W\uparrow_H^G(g).$$

(+)

$\mathcal{O}_r : f_{w,r} \longmapsto w$.

□