Welcome back to abstract linear algebra :)

Let's recall:

**Definition.** Given a group $G$, we say $(\rho, V)$ is a representation if (1) $V$ is a vector space and (2) $\rho : G \to \mathrm{Aut}(V)$ (or $GL(V)$) is a homomorphism.

Last time we discussed how all $\rho(g)$'s are basically just change-of-basis matrices of size $\dim(V)$. (So far all our representations have been finite-dimensional, and that won't change today.) Remember also that our definition of an <u>isomorphism</u> $\phi : V_1 \to V_2$ between representations $\rho_1$ and $\rho_2$ requires that $\phi$ intertwine the actions of $G$ implemented by the $\rho_i$'s: $\phi(\rho_1(g)(v)) = \rho_2(g)\phi(v))$.

**Example.** Let $G = \mathrm{span}\{1, x, x^2, x^3\}$ and $V = \mathbb{C}^2$. Define $\rho : G \to GL_2(\mathbb{C})$ by $x \mapsto \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Then $\rho$ is a representation on $V$.

Question: does there exist a vector $w \in \mathbb{C}^2$ such that $w$ is a common eigenvector amongst all $\rho(g)$ for $g \in G$? Seems like a staunch request... but let's see.

First, note that $\rho(x^2) = \rho(x)^2 = -\mathrm{id}_2$, and by linearity of $\rho$ we only need to find an eigenvector for $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Wow, that got simpler quickly. It's almost like someone chose this example on purpose :)

$\begin{bmatrix} 1 \\ i \end{bmatrix}$ is such an eigenvector with eigenvalue $-i$; let's call it $w$. Hence $\mathrm{span}(w)$ is a stable subspace under the action of $G$ implemented by $\rho$. So here's what we can do: we can restrict the action of $G$ to this subspace and get a representation. (Depending on how fresh your linear algebra is, you might already have already found another linearly independent vector to $w$ which is also stable under this action; we'll use that later!) In some sense this says that there is some basis for $\mathbb{C}^2$ such that the collection $\{\rho(g) : g \in G\}$ acts diagonally simultaneously on $\mathbb{C}^2$ with this basis.

This kind of decomposition of representations into smaller ones is so important we put words to it:

**Definition.** Let $\rho : G \to GL(V)$ be a representation of $G$. Suppose $W$ is a subspace of $V$ (I will write $W < V$, sorry group people) which is $G$-invariant, i.e. $\rho(g)(w) \in W$ for all $g \in G$, $w \in W$. Then $(\rho|_W, W)$ is a representation of $G$ and we say $W$ is a <u>subrepresentation</u> of $V$.

**Example.** (Boring ones) 0 is a subrepresentation. $V$ is one too. All others are called proper because we actually care about those.

**Definition.** The direct sum of two representations of $G$ $(\sigma, S)$ and $(\tau, T)$ is the representation of $G$ on $V = S \oplus T$ given by $\rho(g)(s+t) = \sigma(g)(s) + \tau(g)(t)$. That is, if we reuse notation and write $s := \dim(S)$ and $t := \dim(T)$ we get that

$$\rho(g) = \begin{bmatrix} \sigma(g) & 0 \\ 0 & \tau(g) \end{bmatrix} \in GL_{s+t}(F)$$

where the matrix above is block diagonal.

It turns out that, in our big example above, the span of the vector $u := \begin{bmatrix} 1 \\ -i \end{bmatrix}$ is also stable underneath this action where $u$ has an eigenvalue of $i$. Hence if we choose $(w, u)$ as a basis for $V$, we get $\rho : G \to GL_2(\mathbb{C})$ can be written as $x^j \mapsto \begin{bmatrix} (-i)^j & 0 \\ 0 & i^j \end{bmatrix} = \begin{bmatrix} -i & 0 \\ 0 & j \end{bmatrix}$. Hence $V =\cong \operatorname{span}(u) \oplus \operatorname{span}(w)$.

**Example.** (Exercise (6d)) Let $G$ act on a set $X$. Define a vector space $k[X]$ to be the set of $k$-valued function on $X$ (functions $f : X \to k$). This gives us a $k$-representation $\rho : G \to k[X]$ where $(\rho(g)(f))(x) = f(g^{-1}x)$. Can we find a proper subrepresentation in $k[X]$? (The author uses $F$ but I couldn't stick with it; if we were gonna call $F$ a field we should have done it a long time ago.)

Hmm... my favorite go-to when talking about functions on a space is to look at the constant functions. This serves as a subspace, since 0 is a constant function. We know that $(\rho(g)(f))(x) = f(g^{-1}x)$, but since $f$ is constant we get that $f(g^{-1}\cdot) = f(\cdot)$ and hence is a constant function. Hence the action of $G$ on $k[X]$ is well-defined when restricted to the constant functions on $X$.

In some sense representations of a group without proper subrepresentations are the core elements of that group's representation. Any other representation could be decomposed into these, but these are indecomposable. They're the pivotal things to look at when trying to make claims on a group based on its representation theory. Based on this we make some definitions:

**Definition.** A representation is called irreducible if it contains no proper subrepresentation. It is called completely reducible if it decomposes as a direct sum of irreducible representations.

This "completely reducible" seems interesting. With all the examples we have discussed (fields of characteristic 0, finite-dimensional vector spaces) we have seen stuff that, well, *looks* completely reducible. It turns out that if we add a condition that $G$ is also finite, any such representation of $G$ is completely reducible. We can even consider fields of characteristic not dividing $|G|$. Next lecture someone else will discuss that with you (not me, that's not my vibe).

**Example.** The classification of representations of $S_n$ is a big deal. Hopefully you'll see how terrible it can get but how much Young diagrams can make it a big nicer. It doesn't look like that's on the itinerary for this semester, but I'm dropping some terms here to get you interested. Picture math is cool :)

A standard representation $\rho$ of $G = S_3$ on $V = \mathbb{C}^3$ with basis elements comes from taking a symmetry $\pi \in S_3$ and having it act on the basis elements $\{e_1, e_2, e_3\}$ of $\mathbb{C}_3$, sending $e_i$ to $e_{\pi(i)}$. Question: is there a subrepresentation lurking underneath here?

YES!!!!!!!!!! It's your favorite one whenever you're dealing with symmetries on a finite set. The subspace $\text{span}(e_1+e_2+e_3)$ is *definitely* stable underneath $\rho(\pi)$. But can we find a way to decompose $\rho$ into a direct sum like this? Let's see... let's change our basis from $\{e_1, e_2, e_3\}$ to $\{e_1+e_2+e_3, e_1, e_2\}$. Let's look at the images of the elements of $G$ as matrices over $\mathbb{C}^3$:

$$
\begin{array}{r|l}
e \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & (23) \mapsto \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix} \\
(12) \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} & (123) \mapsto \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \\
(13) \mapsto \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix} & (132) \mapsto \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{bmatrix}
\end{array}
$$

Note that, because $e_1+e_2+e_3$ is an eigenvector with eigenvalue 1, the first column in every matrix is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. But in this basis we've chosen, some element of $e_1 + e_2 + e_3$ is present in how $\rho(\pi)$ acts on $e_1$ and $e_2$ for some $\pi$. That's because $V \ncong \text{span}(e_1 + e_2 + e_3) \oplus \text{span}(e_1, e_2)$. We already kind of knew this; $\text{span}(e_1, e_2)$ is *not* $G$-invariant.

So what *is* $G$-invariant? Oof. These matrices don't look like they're much help. In fact it's not quite clear to me why, but it turns out that the basis we *want* to choose is $\text{span}(e_1 - e_2, e_2 - e_3)$. Once we make this decomposition, we get $V \cong \text{span}(e_1 + e_2 + e_3) \oplus \text{span}(e_1 - e_2, e_2 - e_3)$. It turns out this latter term in the direct sum is irreducible. Exercise: can you prove why?