Representation Theory Notes
for
Iain Gordon's Class
Lecture 6

Abelian Groups

Schur's Lemma : Let V be an irred. rep. of G. Then every G-homomorphism $\phi : V \to V$ (G-endomorphism) is a scalar.
Theorem : All nonzero complex irreducible reps. of an abelian group G have degree 1.

Proof. Let V be a complex irred. rep. of G. Let $g \in G$. Consider the linear transformation $\phi = \rho(g) \cdot V \to V$. It's a G-homomorphism: Let $h \in G$ and $v \in V$.

$$\begin{aligned}\phi(\rho(h)v) &= \rho(g)(\rho(h)v)\\ &= \rho(gh)(v)\\ &= \rho(hg)(v)\\ &= \rho(h)(\rho(g)(v))\\ &= \rho(h)(\phi(v)).\end{aligned}$$

Since V is irred., Schur's Lemma implies that ϕ is scalar. So, $\exists \lambda \in \mathbb{C}$ (depending on g) s.t. $\rho(g)(v) = \phi(v) = \lambda v$. Since g is an arbitrary element of G, this means that each group element acts on V by scalar multiplication.

Every 1-dimensional subspace (line) of V is stable (closed) under scalar multiplication. We just showed every element of G acts on V by scalar multiplication, so every 1-D subspace of V is G-invariant and thus a subrep. We assumed V is irreducible (has no proper subrep.), so V must be 1-dimensional. □

Remark : **(1)** Let $G = C_n = \langle x : x^n = e\rangle$. If (V, ρ) is a complex irred. rep. of G, we just proved that it is 1-dimensional. So, $\rho(x) = \omega$ where $\omega \in \mathbb{C}$. Now, $1 = \rho(e) = \rho(x^n) = \rho(x)^n = \omega^n$. So, ω is an nth root of unity. Thus,

$$\omega = \exp\left(2\pi i \frac{k}{n}\right), k \in \{0, \dots, n-1\}.$$

So, there are at most n irred. complex reps. of G. The nth roots of unity are distinct though, so there are exactly n irred. complex reps. of $G = C_n$.

(2) Each finite abelian group is a product $C_{p_1^{i_1}} \times \cdots \times C_{p_t^{i_t}}$. So, using these ideas, we can describe all complex irred. reps. of finite abelian groups!

(Exercise 4) Let $\rho_j = \exp(2\pi i/p_j^{i_j})$ for $j = 1, \dots, t$. Then, let $C_{p_j^{i_j}} = \langle x_j : x_j^{p_j^{i_j}} = e\rangle$. So, an element of G has the form $(x_1^{a_1}, x_2^{a_2}, \dots, x_t^{a_t})$ with $0 \le a_j \le p_j^{i_j} - 1$ for $1 \le j \le t$.

Now, we get an irreducible rep. of G labeled by a t-tuple of elements $(r_1, r_2, \dots, r_t)$ with $0 \le r_j \le p_j^{i_j} - 1$ by sending $(x_1^{a_1}, x_2^{a_2}, \dots, x_t^{a_t}) \mapsto (\rho_1^{a_1})^{r_1}(\rho_2^{a_2})^{r_2} \cdots (\rho_t^{a_t})^{r_t}$. This produces all possible irred. reps. because x_j has to go to some $\rho_j^{r_j}$. Generators determine where the whole group maps to. These reps. are distinct since the generators are sent to different things in each one.

(Exercise 3) Prove that the image of a G-homomorphism $\phi\colon V \to W$ is a subrepresentation of W.

Sol. Since $\mathrm{Im}\phi$ is a subspace of W, we only need to show that it is G-invariant. So, let $g \in G$ and $x \in \mathrm{Im}\phi$. Then, $x = \phi(v)$ for some $v \in V$. Thus, $\rho_w(g)(x) = \rho_w(g)(\phi(v)) = \phi(\rho_V(g)(v)) \in \mathrm{Im}\phi$. The second equality is true because ϕ is a G-homomorphism. So, $\mathrm{Im}\phi$ is G-invariant and thus a subrep. of W.

Theorem : Let V and W be irred. reps. of G and $\phi : V \to W$ a G-homomorphism.
(i) If V and W are not isomorphic, then ϕ is the zero map.
(ii) If V and W are isomorphic, then ϕ is the zero map or ϕ is an isomorphism.

Proof. Since $\ker\phi$ is a subrep. of V, $\ker\phi$ is 0 or V since V is irreducible.
$\ker\phi = V \iff \phi$ is the zero map; $\ker\phi = 0 \iff \phi$ is injective.
Since $\text{im}\phi$ is a subrep. of W, $\text{im}\phi$ is 0 or W since W is irreducible.
$\text{im}\phi = 0 \iff \phi$ is zero map; $\text{im}\phi = W \iff \phi$ is surjective.
The conclusion follows. □

(Exercises 5,6) $\text{Hom}_G(V, W)$ is an F-vector space under the operations $(\phi + \psi)(v) = \phi(v) + \psi(v)$ and $(\lambda\phi)(v) = \lambda \cdot \phi(v)$ for $\phi, \psi \in Hom_G(V, W)$ and $\lambda \in F$.
Also, $\text{End}_G(V)$ is an F-algebra, and $\text{End}_G(V)$ is a division algebra when V is irreducible.

Sol. The fact that it's a vector space is straightforward, so we just need to show that $\phi, \psi \in Hom_G(V, W)$ and $\lambda, \mu \in F$ implies $\lambda\phi + \mu\psi \in Hom_G(V, W)$. So, we need to show that this G-intertwines.

$$\begin{aligned}(\lambda\phi + \mu\psi)(\rho_V(g)(v)) &= \lambda(\phi(\rho_V(g)(v))) + \mu(\psi(\rho_V(g)(v))) \text{ (def)}\\ &= \lambda(\rho_W(g)(\phi(v))) + \mu(\rho_W(g)\psi(v))\\ &= \rho_W(g)(\lambda(\phi(v))) + \rho_W(g)(\mu(\psi(v))) \text{ (F-linearity)}\\ &= \rho_W(g)((\lambda\phi + \mu\psi)(v)).\end{aligned}$$

Thus, it is G-intertwining, and $\text{Hom}_G(V, W)$ is a vector space.

Sol. Since $\text{End}_G(V) = \text{Hom}_G(V, V)$, we know from **5** that it is a subspace of the F-algebra $\text{End}(V)$ since it is a vector space. We just need to show that it is closed under multiplication of "vectors."
So,

$$\begin{aligned}(\phi\psi)(\rho_V(g)(v)) &= \phi(\psi(\rho_V(g)(v)))\\ &= \phi(\rho_V(g)\psi(v))\\ &= \rho_V(g)(\phi\psi(v)).\end{aligned}$$

Thus, $\phi\psi \in End_G(V)$, and $End_G(V)$ is an F-algebra.
Applying this exercise to $F = \mathbb{C}$ we see that

$$\dim_{\mathbb{C}}\text{Hom}_G(V, W) = \begin{cases} 0 & \text{if } V \ncong W \\ 1 & \text{if } V \cong W. \end{cases}$$

As an example, $\text{End}_G(V) = \mathbb{C}$ when $F = \mathbb{C}$, G is finite, and V is any finite degree irred. rep.

Isotypic Decomposition

(Assumptions: G is finite group, F is a field s.t. $|G|^{-1} \in F$)

Maschke's Theorem (from Lecture 4) tells us that a finite deg. rep. V can be decomposed into a direct sum of irred. reps.: $V = V_1 \oplus \cdots \oplus V_k$. How unique is this? What if $V = V_1' \oplus \cdots \oplus V_j'$. Is there a relationship between these decompositions? The answer is yes! And the isotypic decomposition is what we're looking for.

(Final lemma of talk)

Lemma : Let $\phi \colon V \to W$ be a G-homomorphism. Then V decomposes into a direct sum of G-invariant subspaces $V = U \oplus \ker\phi$ where $U \cong \mathrm{im}\phi \subset W$.

Proof. Recall, $\ker\phi$ is a subrep. of V. By a theorem from Lecture 4 (Theorem 4.1), we can find a G-invariant complement U to $\ker\phi$: $V = U \oplus \ker\phi$. We just need to show that $U \cong \mathrm{im}\phi$. Let $\phi|_U \colon U \to W$ be the restriction of ϕ to U. Since ϕ is a G-homomorphism, so is $\phi|_U$. Also, if $u \in U$, then $u \in \ker\phi|_U \iff u \in \ker\phi \cap U = \{0\}$. This is $\{0\}$ by our definition of U as a complement of $\ker\phi$. Thus, $u = 0$, and $\phi|_U$ is injective.

Next, we'll show that $\phi|_U$ has image $\mathrm{im}\phi$. We will show $\mathrm{im}\phi|_U = \mathrm{im}\phi$. First, note that $\mathrm{im}\phi|_U \subset \mathrm{im}\phi$ For the reverse inclusion, suppose $w \in \mathrm{im}\phi$. Then, for some $v \in V$, $w = \phi(v)$. But, $v = u + k$. So, $w = \phi(u + k) = \phi(u) + \phi(k) = \phi(u) = \phi|_U(u)$. Thus, $w \in \mathrm{im}\phi|_U$. So, $\phi|_U$ is an injective G-homomorphism whose image is $\mathrm{im}\phi$. Thus, $U \cong \mathrm{im}\phi$. □

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Def. Let V be a rep. of G of finite degree. By Maschke's Theorem (Lecture 4, Corollary 4.2), $V = V_1 \oplus \cdots \oplus V_k$ where each V_i is an irred. rep. of G. After re-ordering, we may assume that

$$\begin{aligned} V &= V_1 \oplus \cdots \oplus V_{n_1} \oplus V_{n_1+1} \oplus \cdots \oplus V_{n_2} \oplus \cdots \oplus V_{n_{m-1}+1} \oplus \cdots \oplus V_{n_m} \\ &= V^{I_1} \oplus V^{I_2} \oplus \cdots \oplus V^{I_m} \end{aligned}$$

where $k = n_m$ and $I_1 \cong \{V_1, \ldots, V_{n_1}\}$, $I_2 \cong \{V_{n_1+1}, \ldots, V_{n_2}\}$, $\ldots$, $I_m \cong \{V_{n_{m-1}+1}, \ldots, V_{n_m}\}$. Each I_j is an irred. subrep. of G and $I_i \ncong I_j$ when $i \neq j$. So, we collect together all the $V_i's$ that are isomorphic to one another into m isomorphism classes. The second equality is called an isotypic decomposition of V and each V^{I_j} is called an isotypic component.