Abelian Groups/Isotypic Decomposition

Wednesday, September 30, 2020 9:44 AM

Schur's Lemma: V irred. rep. of G

Then, every G-homomorphism $\phi : V \to V$ is scalar

Thm. All nonzero complex irred. reps. of an ab. gp. G have deg 1.

PF: V, let $g \in G$. $\phi = \rho(g) \cdot V \to V$.

Let $h \in G$, $v \in V$. $\phi(\rho(h)(v)) = \rho(g)(\rho(h)(v))$

$$= \rho(hg)(V)$$
$$= \rho(h)(\rho(g)(v))$$
$$= \rho(h)(\phi(v))$$

So, $\exists \lambda \in \mathbb{C}$ s.t. $\rho(g)(v) = \phi(v) = \lambda v$.

W subspace of $V \Rightarrow V$ is 1-D (or has deg 1). $\square$

1) Let $G = C_n = \langle x : x^n = e \rangle$. Let (V, ρ).

$\rho(x) = w$ where $w \in \mathbb{C}$

$$1 = \rho(e) = \rho(x^n) = \rho(x)^n = w^n$$

$$w = \exp\left(2\pi i \frac{k}{n}\right), \quad k \in \{0, \dots, n-1\}$$

2) $G = C_{p_1^{a_1}} \times \cdots \times C_{p_t^{a_t}}$

Let $\rho_j = \exp(2\pi i / p_j^{a_j})$ for $j = 1, \dots, t$.

$C_{p_j^{a_j}} = \langle x_j ; x_j^{p_j^{a_j}} = e \rangle$.

$(x_1^{a_1}, x_2^{a_2}, \dots, x_t^{a_t})$, $\quad 0 \leq a_j \leq p_j^{a_j} - 1$

$(r_1, r_2, \dots, r_t)$; send $(x_1^{a_1}, x_2^{a_2}, \dots, x_t^{a_t}) \longmapsto (\rho_1^{a_1})^{r_1} (\rho_2^{a_2})^{r_2} \cdots (\rho_t^{a_t})^{r_t}$

G-homomorphism $\phi : V \to W$ then $\operatorname{Im} \phi$ is a subrep of W

Let $g \in G$ and $x \in \operatorname{Im} \phi$.

Then, $x = \phi(v)$ for some $v \in V$.

Thus, $\rho_W(g)(x) = \rho_W(g)(\phi(v)) = \phi(\rho_V(g)(v)) \in \operatorname{Im} \phi$

Thm: Let V, W be irred. reps. of G and $\phi : V \to W$ is a G-HM.

(i) If $V \not\cong W$, then ϕ is zero map.

(i) If $V \not\cong W$, then ϕ is zero map.

(ii) If $V \cong W$, then ϕ " " or ϕ is an isomorphism.

Pf: $\operatorname{Ker}\phi$ is a subrep of V, so $\operatorname{Ker}\phi$ is 0 or V.

$\operatorname{Ker}\phi = V \iff \phi$ is zero map; $\operatorname{Ker}\phi = 0 \iff \phi$ is injective

$\operatorname{Im}\phi$ is a subrep of W, so $\operatorname{Im}\phi$ is 0 or W.

$\operatorname{Im}\phi = 0 \iff \phi$ is zero map; $\operatorname{Im}\phi = W \iff \phi$ is surjective.

Ex 5) $\operatorname{Hom}_G(V,W)$ is an F-vector space under

$$(\phi + \psi)(v) = \phi(v) + \psi(v) \quad \text{and} \quad (\lambda\phi)(v) = \lambda \cdot \phi(v)$$

$$\begin{aligned}(\lambda\phi + \mu\psi)(\rho_V(g)(v)) &= \lambda(\phi(\rho_V(g)(v))) + \mu(\psi(\rho_V(g)(v))) \\ &= \lambda(\rho_W(g)(\phi(v))) + \mu(\rho_W(g)(\psi(v))) \\ &= \rho_W(g)(\lambda(\phi(v))) + \rho_W(g)(\mu(\psi(v))) \\ &= \rho_W(g)((\lambda\phi + \mu\psi)(v))\end{aligned}$$

Ex 6) $\operatorname{End}_G(V)$ is also an F-algebra

$$(\phi\psi)(\rho_V(g)(v)) = \phi(\rho_V(g)\psi(v))$$

$$= \rho_V(g)(\phi\psi(v))$$

$F = \mathbb{C}$

$$\dim_{\mathbb{C}} \operatorname{Hom}_G(V,W) = \begin{cases} 0 & \text{if } V \not\cong W \\ 1 & \text{if } V \cong W \end{cases}$$

Let V be a rep. of G of finite deg.

$V = V_1 \oplus \cdots \oplus V_k$ where each V_i is irred. rep of G.

We may assume

$$V = \overbrace{V_1 \oplus \cdots \oplus V_{n_1}}^{I_1} \oplus \overbrace{V_{n_1+1} \oplus \cdots \oplus V_{n_2}}^{I_2} \oplus \cdots \oplus \overbrace{V_{n_{m-1}+1} \oplus \cdots \oplus V_{n_m}}^{I_m}$$

Isotypic Decomposition $\rightarrow$ $V = V^{I_1} \oplus V^{I_2} \oplus \cdots \oplus V^{I_m}$.

Isotypic Component

$V = V_1 \oplus \cdots \oplus V_k$ How unique is this?

$V = V_1' \oplus \cdots \oplus V_k'$

Lemma: Let $\phi : V \to W$ be a G-HM. Then V decomposes into a direct sum of G-invariant subspaces $V = U \oplus \ker\phi$ where $U \cong \operatorname{Im}\phi \subseteq W$.

where $U \cong \mathrm{Im}\,\phi \subseteq W$.

PF: $V = U \oplus \ker\phi$. Let $\phi|_U : U \to W$

$\phi|_U$ is a G-HM.

Suppose $u \in U$. Then, $u \in \ker \phi|_U \Leftrightarrow u \in \ker\phi \cap U = \{0\}$.

So, $u = 0$, and $\phi|_U$ is injective.

We know $\mathrm{Im}\,\phi|_U \subset \mathrm{Im}\,\phi$. Suppose $w \in \mathrm{Im}\,\phi$. Then $w = \phi(v)$.

But $v = u + k$, $w = \phi(v) = \phi(u+k) = \phi(u) + \phi(k) = \phi(u) \in \mathrm{Im}\,\phi|_U$.

$U \cong \mathrm{Im}\,\phi$.