Representation Theory Notes
for
Iain Gordon's Class
Lecture 10

Proof of the Orthogonality of Characters

Theorem 7.8: Suppose V and W are irreducible representations. Then,
(i) $\langle \chi_V \mid \chi_V \rangle = 1$
(ii) $\langle \chi_V \mid \chi_W \rangle = 0$ if $V \ncong W$.

Before proving this theorem, we need a sublemma.

Sublemma: Let $f\colon V \to W$ be **any** linear transformation, and set

$$f^\circ = \frac{1}{|G|} \sum_{g \in G} \rho_w(g^{-1}) f \rho_v(g) \colon V \to W.$$

Then, *(a)* f° is a G-homomorphism;
(b) if $V \ncong W$, then $f^\circ = 0$;
(c) if $V = W$ and $\rho_w = \rho_v$, then $f^\circ = \frac{\mathrm{Tr}(f)}{\dim V} \cdot \mathrm{Id}_v$.

Proof. (a) Let $h \in G$. Then,

$$\begin{aligned}
\rho_w(h^{-1}) f^\circ \rho_v(h) &= \frac{1}{|G|} \sum_{g \in G} \rho_w(h^{-1}) \rho_w(g^{-1}) f \rho_v(g) \rho_v(h) \\
&= \frac{1}{|G|} \sum_{g \in G} \rho_w((gh)^{-1}) f \rho_v(gh) \\
&= \frac{1}{|G|} \sum_{g \in G} \rho_w(g^{-1}) f \rho_v(g) \\
&= f^\circ.
\end{aligned}$$

(b) Recall Theorem 5.7(i) (Lecture 6): If $V \ncong W$ and $\phi\colon V \to W$ is a G-homomorphism, then ϕ is the zero map. By (a), f° is a G-homomorphism, so $V \ncong W$ implies $f^\circ = 0$.

(c) By Theorem 5.7(ii), $f^\circ = c \cdot \mathrm{Id}_V$ where c is some constant. We need to show that $c = \mathrm{Tr(f)}/\dim V$. So, taking traces gives us

$$\begin{aligned}
c \cdot \dim V &= \mathrm{Tr}(c \cdot \mathrm{Id}_V) \\
&= \mathrm{Tr}(f^\circ) \\
&= \frac{1}{|G|} \sum_{g \in G} \mathrm{Tr}(\rho_w(g^{-1}) f \rho_v(g)) \\
&= \frac{1}{|G|} \sum_{g \in G} \mathrm{Tr}(\rho_v(g^{-1}) f \rho_v(g)) \\
&= \frac{1}{|G|} \sum_{g \in G} \mathrm{Tr}(f) \\
&= \mathrm{Tr}(f).
\end{aligned}$$

Dividing by $\dim V$ gives us $c = \mathrm{Tr}(f)/\dim V$, and the conclusion follows. □

With this sublemma, we will now prove Theorem 7.8.

Proof. Choose a basis for V and W so that any linear mappings are given by matrices. Then,

$$\begin{aligned}\langle \chi_W \mid \chi_V \rangle &= \frac{1}{|G|}\sum_{g\in G}\overline{\chi_W(g)}\chi_V(g) \\ &= \frac{1}{|G|}\sum_{g\in G}\chi_W(g^{-1})\chi_V(g) \quad \text{(Theorem 7.5(c))} \\ &= \frac{1}{|G|}\sum_{g\in G}\mathrm{Tr}(\rho_w(g^{-1}))\mathrm{Tr}(\rho_v(g)) \\ &= \frac{1}{|G|}\sum_{g\in G}\sum_{s,t}\rho_w(g^{-1})_{ss}\rho_v(g)_{tt} \text{ where } 1\le s\le \dim W, 1\le t\le \dim V.\end{aligned}$$

Now, we are going to use the sublemma we proved earlier. In particular, recall that in the statement of the sublemma, f can be any linear transformation. So, we have

$$(f^\circ)_{st} = \frac{1}{|G|}\sum_{g\in G}\sum_{k,l}\rho_w(g^{-1})_{sk}f_{kl}\rho_v(g)_{lt}. \quad (\star)$$

Now, we consider the two parts of the Theorem. First, suppose $V \ncong W$ (part (ii)). By part (b) of the sublemma, $f^\circ = 0$. Thus, $(f^\circ)_{st} = 0$ for all s, t (LHS is just 0). Now, we pick f to be the matrix such that $f_{kl} = 1$ if $k = s$ and $l = t$ and $f_{kl} = 0$ otherwise.

$$f_{kl} = \begin{cases} 1 & \text{if } k = s \text{ and } l = t \\ 0 & \text{otherwise.}\end{cases}$$

In this case, $(\star)$ becomes

$$0 = \frac{1}{|G|}\sum_{g\in G}\rho_w(g^{-1})_{ss}\rho_v(g)_{tt}$$

because every other term is zero when we sum over k, l. Since the LHS is zero, we can sum over s, t to get

$$0 = \frac{1}{|G|}\sum_{g\in G}\sum_{s,t}\rho_w(g^{-1})_{ss}\rho_v(g)_{tt}$$

The RHS is just $\langle \chi_W \mid \chi_V \rangle$, so it follows that $\langle \chi_W \mid \chi_V \rangle = 0$.

For part (i), suppose $V = W$ and $\rho_v = \rho_w$. Using the sublemma, $(\star)$ becomes

$$\delta_{st}\frac{\mathrm{Tr}(f)}{\dim V} = \frac{1}{|G|}\sum_{g\in G}\sum_{k,l}\rho_v(g^{-1})_{sk}f_{kl}\rho_v(g)_{lt}$$

Note that $s \neq t$ implies the LHS is zero as $\delta_{st} = 0$. If $s = t$, then the LHS is $\mathrm{Tr}(f)/\dim V$. If we define as f the same as before, then $\mathrm{Tr}(f) = 1$ since $k = s = t = l$ occurs on the diagonal. Summing over s, t gives

$$\langle \chi_V \mid \chi_V \rangle = \frac{1}{|G|}\sum_{g\in G}\sum_{s,t}\rho_v(g^{-1})_{ss}\rho_v(g)_{tt} = \sum_{s=t}\frac{1}{\dim V} = \frac{\dim V}{\dim V} = 1.$$

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Corollary 7.13: $V \cong W \iff \chi_V = \chi_W$.

Proof. $(\implies)$ Theorem 7.3(i) (Middle of Lecture 8)

($\Longleftarrow$) Take decompositions $V \cong I_1^{m_1} \oplus \cdots \oplus I_k^{m_k}$ and $W \cong I_1^{n_1} \oplus \cdots \oplus I_k^{n_k}$ where $\{I_1, \ldots, I_k\}$ is a complete set of irreducible representations for G.

Now, by Corollary 7.9, $\langle \chi_V \mid \chi_{I_j} \rangle = m_j$ and $\langle \chi_W \mid \chi_{I_j} \rangle = n_j$. Since $\chi_V = \chi_W$, it follows that $m_j = n_j$. So, $V \cong W$. $\square$

Character Tables

Definition 8.1: A **class function** on G is a function $f\colon G \to \mathbb{C}$ such that $f(g) = f(xgx^{-1})$ for all $x, g \in G$. In other words, f is a complex-valued function on G that is constant on conjugacy classes.

Examples:

1) Any character χ_V is a class function.

2) Let $Cl(g)$ be the conjugacy class of g in G. Then, the following is a class function

$$ch_{Cl(g)}(x) = \begin{cases} 1 & \text{if } x \in Cl(g) \\ 0 & \text{if } x \notin Cl(g). \end{cases}$$

3) The following is a class function if and only if $g \in Z(G)$, the center of the group G:

$$ch_g(x) = \begin{cases} 1 & x = g \\ 0 & x \neq g. \end{cases}$$