Representation Theory Notes
for
Iain Gordon's Class
Lectures 13/14

## Lifting Representations

**Corollary 9.5:** If $N \triangleleft G$, then there exist characters $\chi_1, \ldots, \chi_s$ such that $N = \bigcap_{i=1}^s \mathrm{Ker}\chi_i$.

*Proof.* Let $\hat{\chi}_1, \ldots, \hat{\chi}_s$ be the irreducible characters of $G/N$ and lift them to $\chi_1, \ldots, \chi_s$ characters of $G$. Then, $N \subset \mathrm{Ker}\chi_i$, so $N \subset \bigcap_1^s \mathrm{Ker}\chi_i$.

Now, choose $g \in \mathrm{Ker}\chi_i$. By Lemma 9.2, $\chi_i(g) = \chi_i(e)$ for all $i \in \{1, \ldots, s\}$. By definition, this implies $\hat{\chi}_i(gN) = \hat{\chi}_i(eN)$ for all $i$. Now, the irreducible characters form a basis for all class functions of $G/N$, so it follows that $\chi(gN) = \chi(eN)$ for *any* class function $\chi$ of $G/N$. Take $\chi$ to be the class function $\chi_{\mathrm{Cl}(eN)}$. Thus, $gN = eN$, so $g \in N$. Therefore, $\bigcap_1^s \mathrm{Ker}\chi_i \subset N$, and the claim follows. □

## Character Table for $A_5$

The conjugacy classes of $S_5$ are labeled by cycle type:

$$\mathbf{e}, (12), \mathbf{(123)}, \mathbf{(12)(34)}, (1234), (123)(45), \mathbf{(12345)}.$$

Only the bolded types belong to $A_5$, but it is possible that elements conjugate in $S_5$ are not conjugate in $A_5$. This isn't a problem for the first three.

Suppose $x$ is an odd permutation. If $h = x(123)x^{-1}$, then $x(45)$ is even and

$$(x(45))(123)(45)x^{-1} = x(123)x^{-1} = h.$$

If $h = x(12)(34)x^{-1}$, then $x(12)$ is even and

$$(x(12))(12)(34)(12)x^{-1} = x(34)(12)x^{-1} = x(12)(34)x^{-1} = h.$$

So, $h$ and $g$ are conjugate in $A_5$. It is obvious in the case of $e$. There is a problem for $g = (12345)$. The centralizers of $g$ in each group are

$$C_{S_5}(g) = \langle g \rangle \leq A_5; \quad C_{A_5}(g) = \langle g \rangle.$$

Now, the order of the conjugacy class is the order of the group divided by the order of the centralizer, so

$$|\mathrm{Cl}_{A_5}(g)| = \frac{|A_5|}{|C_{A_5}(g)|} = \frac{\frac{1}{2}|S_5|}{|C_{S_5}(g)|} = \frac{1}{2}|\mathrm{Cl}_{S_5}(g)|.$$

Thus, the conjugacy classes split, so there are five conjugacy classes in $A_5$:

$$e, (123), (12)(34), (12345), (13245).$$

So, $A_5$ has five irreducible representations (by a previous lemma). From a previous exercise (2 in the Lec 8/9 wkst), we have $\mathbb{C}^5$ is a representation of $S_5$, so we get a representation of $A_5$ by restriction. It has form $\chi_{\mathbb{C}^5} = \chi_1 + \chi_V$. Thus, $\chi_V$ has character $\chi_{\mathbb{C}^5} - \chi_1$. So, $\chi_V : 4 \quad 1 \quad 0 \quad -1 \quad -1$. Then,

$$\langle \chi_V \mid \chi_V \rangle = \frac{1}{60}(4^2 + 20 \cdot 1^2 + 15 \cdot 0^2 + 12 \cdot (-1)^2 + 12 \cdot (-1)^2) = 1.$$

Thus, $\chi_V$ is irreducible; call $\chi_V = \chi_2$. We want to find more, so let's look at pairs of numbers between 1 and 5. $W$ has basis $e_{\{i,j\}}$ for $1 \leq i, j \leq 5$. This produces a representation of $S_5$, so it also produces a representation of $A_5$. Now, consider the smaller pieces

$$E = span\{e_{\{i,j\}} - e_{\{j,i\}}\}$$

and

$$S = span\{e_{\{i,j\}} + e_{\{j,i\}}\}$$

where $E$ has the 10-dimensional basis $\{e_{\{i,j\}} - e_{\{j,i\}} : 1 \leq i < j \leq 5\}$ since it is symmetric and $S$ has the 15-dimensional basis $\{e_{\{i,j\}} + e_{\{j,i\}} : 1 \leq i \leq j \leq 5\}$.

**Lemma:** $\chi_E : 10 \quad 1 \quad -2 \quad 0 \quad 0$ and $\chi_S : 15 \quad 3 \quad 3 \quad 0 \quad 0$.

*Sketch:* For $\chi_S$, $g(e_{\{i,j\}} + e_{\{j,i\}}) = e_{\{gi,gj\}} + e_{\{gj,gi\}}$, so we just need to count the number of $(i,j)$ that remain unchanged for each conjugacy class representative since we are calculating the trace. The same method works for $\chi_E$ except we have to be aware of negative values.

We can calculate the following:

$$\langle \chi_S \mid \chi_1 \rangle = 2$$
$$\langle \chi_S \mid \chi_2 \rangle = 2$$

So, $\chi_3 = \chi_S - 2\chi_1 - 2\chi_2$ is a character with $\chi_3 : 5 \quad -1 \quad 1 \quad 0 \quad 0$. Then,

$$\langle \chi_3 \mid \chi_3 \rangle = \frac{1}{60}(5^2 + 20(-1)^2 + 15(1^2) + 0 + 0) = 1.$$

So, we have found three irreducible representations $\chi_1, \chi_2$, and $\chi_3$, but we still need to find $\chi_4$ and $\chi_5$. First, $\chi_1(e) + \chi_2(e) + \chi_3(e) = 10$, so $\chi_4(1) + \chi_5(1) = 6$. Also, the sum of the squares is $|A_5| = 60$, so $1^2 + 4^2 + 5^2 + \chi_4(1)^2 + \chi_5(1)^2 = 60$. We get the following system of equations:

$$\chi_4(1) + \chi_5(1) = 6$$
$$\chi_4(1)^2 + \chi_5(1)^2 = 18.$$

The only solution is $\chi_4(1) = \chi_5(1) = 3$. This gives us the first column of the character table. The rest of the table requires four boards of similar brute force calculations, so we will skip to the answer (note: the simplicity of $A_5$ falls out as a corollary during the work).

A character table of $A_5$:

| | e | (123) | (12)(34) | (12345) | (13245) |
|---|---|---|---|---|---|
| | 1 | 20 | 15 | 12 | 12 |
| $\chi_1$ | 1 | 1 | 1 | 1 | 1 |
| $\chi_2$ | 4 | 1 | 0 | -1 | -1 |
| $\chi_3$ | 5 | -1 | 1 | 0 | 0 |
| $\chi_4$ | 3 | 0 | -1 | $\frac{1+\sqrt{5}}{2}$ | $\frac{1-\sqrt{5}}{2}$ |
| $\chi_5$ | 3 | 0 | -1 | $\frac{1-\sqrt{5}}{2}$ | $\frac{1+\sqrt{5}}{2}$ |

## Induced Representations

We want to compare representations of $G$ and of $H$ when $H < G$.

**Defintion:** Given $(V, \rho)$ of $G$, we can consider the reprsentation of $H$ given by restriction ($\text{res}_H^G V$) $H \to G \to GL(V)$.

**Lemma:** Let $\chi_V$ be the character of $V$. Then, $\chi_{\text{res}_H^G V} = \chi_V|_H$.

*Proof.* Clear. ☐

We used this idea previously when we began with a character for $S_5$ and restricted to $A_5$. Next, we wish to begin with a representation of a subgroup and produce a representation of the group.

**Definition:** Let $(W, \rho_W)$ be a representation of $H$. The induced representation $\mathrm{Ind}_H^G W$ is the space

$$\{f : G \to W \mid f(gh) = \rho_W(h)^{-1} f(g) \quad \forall h \in H, \forall g \in G\}.$$

**Lemma:** $\mathrm{Ind}_H^G W$ is a representation of $G$.

*Proof.* It is straightforward to show that $\mathrm{Ind}_H^G W$ is a vector subspace of the space of functions $G \to W$, so we are going to skip it.

The action of $G$ is given by $(g \cdot f)(k) = f(g^{-1}k)$ for $g, k \in G$ and $f \in \mathrm{Ind}_H^G W$. We need to show that this action is linear on $\mathrm{Ind}_H^G W$. First, it is well-defined (i.e. closed, action stays in $\mathrm{Ind}_H^G W$).

$$(g \cdot f)(kh) = f(g^{-1}kh) = \rho_W(h)^{-1}(f(g^{-1}k)) = \rho_W(h)^{-1}(g \cdot f(k)), \quad \forall g, k \in G, \forall h \in H, \forall f \in \mathrm{Ind}_H^G W.$$

Next, it is a homomorphism:

$$((g_1 \cdot g_2) \cdot f)(k) = f((g_1 g_2)^{-1}k) = f(g_2^{-1} g_1^{-1} k) = (g_2 \cdot f)(g_1^{-1}k) = (g_1 \cdot (g_2 \cdot f))(k)$$

Finally, it is linear:

$$\begin{aligned}
(g \cdot (\lambda_1 f_1 + \lambda_2 f_2))(k) &= (\lambda_1 f_1 + \lambda_2 f_2)(g^{-1}k) \\
&= \lambda_1(f_1(g^{-1}k)) + \lambda_2(f_2(g^{-1}k)) \\
&= \lambda_1((g \cdot f_1)(k)) + \lambda_2((g \cdot f_2)(k)) \\
&= (\lambda_1(g \cdot f_1) + \lambda_2(g \cdot f_2))(k).
\end{aligned}$$

Therefore, this is a linear action on $\mathrm{Ind}_H^G W$. $\square$

**Examples:** (i) $H = G$; $\mathrm{Ind}_H^G W \cong W$. We just send $f$ to $f(e_G)$.
(ii) $H = \{e\}$; $\mathrm{Ind}_H^G \mathbb{C} = \mathbb{C}[G]$, the regular representation.
(iii) $\mathrm{Ind}_{A_4}^{A_5} \mathbb{C} \cong I_1 \oplus I_2$ (see Exercise 3 in Lec 13/14 wkst)