SWAG lecture 5

Wednesday, September 23, 2020 9:50 AM

Recall

Def:

(1) irreducible

a representation is called irreducible if it contains no proper subrep.

(2) completely reducible

It's called ↓ if it decomposes as a direct sum of irreducible subrep.

Example: The left regular rep.

suppose G is a finite group, F is a field.

$F[G]$: vector space of F-valued functions on G. it has vector space basis $\{e_g : g \in G\}$.

$e_g =$ $e_g(h) = \begin{cases} 1 & h = g \\ 0 & \text{otherwise.} \end{cases}$

"delta function".

G acts on basis by

$\rho(g) e_h = e_{gh}$.

left translation by g.

extends by linearity to $F[G]$ $\lambda_h \in F$.

$\rho(g) \left(\sum_{h \in G} \lambda_h e_h \right) = \sum_{h \in G} \lambda_h \rho(g) e_h = \sum_{h \in G} \lambda_h e_{gh}$

$F[G]$ is a rep of G, called the left regular rep.

right regular rep.

$\rho(g)(e_h) = e_{hg^{-1}}$

e_{hg}.

Example: $G = C_2$, $F = \mathbb{F}_2 = \{0, 1\}$

$= \{1, x\}$

$x^2 = 1$.

$F[G]$ is not completely reducible.

Let $w = e_1 + e_x$. $W = \text{span}(w)$.

W is G-invariant.

$\rho(x)(e_1 + e_x) = (e_{x \cdot 1} + e_{x^2})$

$= e_x + e_1 = e_1 + e_x$

check there is no other G-invariant subspaces.

$\text{span}\{e_1, e_x\} = \{0, e_1, e_x, e_1 + e_x\}$

$F[G]$ is not completely reducible.

Exp 2: $G = C_n$, $F = \mathbb{C}$, $V = \mathbb{C}[C_n]$ is completely reducible.

$\dim V = n$.

Let $\varphi = \exp\left(\frac{2\pi i}{n}\right)$.

define $E_k = \sum_{j=0}^{n-1} \varphi^{kj} e_{x^j} \in \mathbb{C}[G]$

$0 \le k \le n-1$.

$\rho(x)(E_k) = \sum_{j=0}^{n-1} \varphi^{kj} e_{x \cdot x^j}$

$= \sum_{j=0}^{n-1} \varphi^{kj} e_{x^{j+1}}$

$= \left[\sum_{j=0}^{n-1} \varphi^{(j+1)k} e_{x^{j+1}} \right] \cdot \varphi^{-k}$

$= E_k \cdot \varphi^{-k}$

$\Rightarrow E_k$ spans a 1-D subrep of $\mathbb{C}[C_n]$

$\mathbb{C}[C_n] = \bigoplus_k \mathbb{C} E_k$

$\Rightarrow \mathbb{C}[C_n]$ is completely reducible.

G-homo & Shur's Lemma.

Def: Let (V^1, ρ^1) and (V^2, ρ^2) be rep of G over F. A G-homomorphism from (V^1, ρ^1) to (V^2, ρ^2) is an F-linear mapping $\phi: V^1 \to V^2$, which intertwines the action of G.

i.e. $\phi(\rho^1(g) v) = \rho^2(g) \phi(v)$ for all $g \in G$, $v \in V^1$

⇓

$$\begin{array}{ccc} V^1 & \xrightarrow{\rho^1(g)} & V^1 \\ \phi \downarrow & & \downarrow \phi \\ V^2 & \xrightarrow{\rho^2(g)} & V^2 \end{array}$$

the diagram commutes.

(1) Denote $\text{Hom}_G(V^1, V^2) = \{\phi : V^1 \to V^2 : \phi \text{ is } G\text{-homo}\}$

$\text{End}_G(V^1) = \{\phi : V^1 \to V^1, \phi \text{ is } G\text{-homo}\}$

Remark:

(1). An invertable G-homomorphism ⇕ G-isomorphism

(2) usually we drop ρ^1 and ρ^2 from notation.

$\phi(gv) = g\phi(v)$

Lemma: Let $\phi: V \to W$ be a G-homo.

(1) $\ker\phi$ and $\text{Im}\,\phi$ is a subrep of V and W respectively.

Pf: it suffices to show $\ker\phi$ and $\text{Im}\,\phi$ are G-invariant.

(1) if $k \in \ker\phi$, $g \in G$,

$\phi(\rho_V(g)(k)) = \rho_W(g)(\phi(k)) = \rho_W(g)(0)$

ϕ is G-homo. $= 0$. by linearility.

$\Rightarrow \rho_V(g)(k) \in \ker\phi$.

$\Rightarrow \ker\phi$ is G-invariant

(2) if $l \in \text{Im}\,\phi$, $\Rightarrow \exists\ k$ s.t

$l = \phi(k)$

$\rho_W(g)(l) = \rho_W(g)(\phi(k)) = \phi(\rho_V(g)(k))$

$\Rightarrow \rho_W(g)(l) \in \text{Im}\,\phi$.

$\Rightarrow \text{Im}\,\phi$ is G-invariant.

Lemma: (shur's Lemma)

Assume the field F is algebraically closed, G is a finite group. Let φ, ρ be two irreducible rep of G, and $T \in \text{Hom}_G(\varphi, \rho)$, Then either T is 0 or T is invertable.

So, (1). If φ is not isomorphic to ρ, then $\text{Hom}_G(\varphi, \rho) = 0$;

(2). if $\varphi = \rho$. Then $T = \lambda I$ with $\lambda \in F$.

Pf: $\varphi: G \to GL(V)$ $\rho: G \to GL(W)$

$T: V \to W$,

if $T \ne 0$, then, by last Lemma, $\ker T$ and $\text{Im}\,T$ are irreducible,

$\Rightarrow \ker T$ is either 0 or V,

$\text{Im}\,T$ is either 0 or W.

since T is not 0, $\Rightarrow$ $\ker T \ne V$, $\text{Im}\,T \ne 0$.

T is an isomorphism, $\ker T$ is 0, $\text{Im}\,T = W$.

(2): Since F is algebraically closed. there $\exists$ a λ and $v \in V$ s.t. $Tv = \lambda v$

eigenvalue. $\Rightarrow \lambda I - T$ is in $\text{Hom}_G(\varphi, \rho)$

since. $\lambda I - T$ is not invertable, $\ker(\lambda I - T) \ne 0$,

$(\lambda I - T) g = 0$.

$(\lambda I - T) 0 = 0$.

$\Rightarrow \lambda I - T = 0$.

$T = \lambda I$.

λ is an eigenvalue of T,

Cor let G be an abelian group,

Then any irreducible rep of G has degree one.

Let $\varphi: G \to GL(V)$, be an irreducible rep,

pick a $h \in G$, we can set $T = \varphi_h$,

for any $g \in G$,

$T\varphi_g = \varphi_h \cdot \varphi_g = \varphi_{hg} = \varphi_{gh} = \varphi_g \varphi_h = \varphi_g T$.

By shur's Lemma, $T = \varphi_h = \lambda_h \cdot I$.

Let v be a non-zero vector in V, $k \in F$,

$\varphi_h(kv) = \lambda_h kv \in F[v]$

$\Rightarrow F[v]$ is G-invariant. subspace,

since $T = \varphi_h$ is irreducible

$V = Fv$. $\Rightarrow \dim V = 1$.