SWAG lecture 8

Wednesday, October 21, 2020 9:58 AM

Def: irr G- rep $I_1, I_2 \cdots I_n$ form a complete set of non-isomorphic irr G-rep if every G- rep is isomorphic to some I_j, and no two of $I_1, \cdots I_n$ are isomorphic.

Character Theory.

In this section, G is a finite group, $F = \mathbb{C}$.

Def: Let $(V; \rho)$ be a finite degree G-rep of G, and choose a basis of V, then write the rep ρ in terms of matrices. The character of V, written as χ_V is the function

$$\chi_V : G \longrightarrow \mathbb{C}.$$
$$g \to Tr(\rho(g))$$

we say that χ_V is irr if ρ is irr.

Rmk:

(a) χ_V is independent of the choice of basis.

Pf: if ρ' is given by another basis, $\Rightarrow \rho$ and ρ' are isomorphic $\Rightarrow \exists$ a invertible matrix X such that

$$\rho'(g) = X\rho(g)X^{-1}.$$

$$Tr(\rho'(g)) = Tr(X\rho(g)X^{-1})$$
$$= Tr(\rho(g))$$

$Tr(AB) = Tr(BA)$

Prop.

(1) if $V \cong W$, then $\chi_V = \chi_W$

(2) if $g, h \in G$ are conjugate, then $\chi_V(g) = \chi_V(h)$

Pf: (1) pick a basis for V and W, let $\phi : V \to W$ is an isomorphism, written in terms of these basis,

Then, $\rho_W(g) = \phi\, \rho_V(g)\, \phi^{-1} \quad \forall g \in G$

$$\chi_W(g) = Tr(\rho_W(g)) = Tr(\phi \rho_V(g) \phi^{-1})$$
$$= Tr(\rho_V(g)) = \chi_V(g)$$

(2) $\exists\ x \in G$, s.t. $g = xhx^{-1}$

$$\chi_V(g) = \chi_V(h)$$

Exp:

(1) $C_3 = \{e, x, x^2\}$,

C_3 has 3 irreducible reps, each of them is 1-D,

$$\rho_i(x^j) = w^{ij} \qquad w = \exp\left(\frac{2\pi i}{3}\right)$$

$$\chi_{\rho_0}(e) = Tr(\underbrace{\rho_0(e)}_{1-D}) = \rho_0(e) = 1$$

$$\chi_{\rho_1}(x) = Tr(\rho_1(x)) = w$$

	e	x	x^2
χ_{ρ_0}	1	1	1
χ_{ρ_1}	1	w	w^2
χ_{ρ_2}	1	w^2	w.

(b) S_3 :

$$\varphi_{(1\,2)} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \varphi_{(1\,2\,3)} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$I_{3\times3}$.

$$\chi_\varphi(Id) = 3$$
$$\chi_\varphi((1\,2)) = 1.$$
$$\chi_\varphi((1\,2\,3)) = 0.$$

(c). $V = \mathbb{C}[G] = \{\sum_{x \in G} c_x \cdot x \mid c_x \in \mathbb{C}\}$

regular rep:

$$L_g(h) = g \cdot h$$

$$\chi_{reg}(g) = \begin{cases} |G|, & g = e \\ 0 & \text{otherwise.} \end{cases}$$

Pf: G is finite $\Rightarrow G = \{g_1, \cdots g_n\}$

$n = |G| \qquad L_g g_j = g \cdot g_j$

$\Rightarrow$ Let $[L_g]$ denote the matrix of L_g with respect to the basis of G. in this order,

then $[L_g]_{ij} = \begin{cases} 1 & g_i = g \cdot g_j \\ 0 & \text{else.} \end{cases}$

$$= \begin{cases} 1 & g = g_i g_j^{-1}. \\ 0 & \text{else.} \end{cases}$$

In particular $[L_g]_{ii} = \begin{cases} 1 & g = 1 \\ 0 & \text{else.} \end{cases}$

$$\chi_L(g) = Tr[L_g]$$
$$= \begin{cases} n = |G|. & g = 1 \\ 0 & g \neq 1. \end{cases}$$

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Thm Let V be a G-rep of G, $g \in G$,

(a) $\chi_V(e) = \dim V$

(b) $\chi_V(g)$ is a sum of roots of unity

(c) $\chi_V(g^{-1}) = \overline{\chi_V(g)}$

(d) $\chi_{V \oplus W} = \chi_V + \chi_W$

(e) $\overline{\chi_V}$ is also a character.

$$\overline{\chi_V}(g) := \overline{\chi_V(g)}$$

Pf: (a) $\chi_V(e) = Tr(\underbrace{\rho_V(e)}_{Id}) = \dim V$

(b) Let $H = \langle g \rangle$ be the subgroup generated by g, and restrict $\rho_V|_H \Rightarrow$ a rep of H.

H is an abelian, cylic group,

consider V as a rep of H, then decompose

$V = V_1 \oplus \cdots \oplus V_k \qquad V_i$ irr

since every irr rep of abelian group is 1-dim

$\Rightarrow V_i$ is one dimension.

if $v_i \in V_i$, then

$$\rho_V(g)(v_i) = w_i \cdot v_i,$$
$$w_i^{|H|} = 1.$$

more over, $(v_1, \cdots v_k)$ is a basis of V,

wrt to this basis,

$$\rho(g) = \begin{pmatrix} w_1 & & 0 \\ & w_2 & \\ 0 & & \ddots \; w_k \end{pmatrix}$$

$$\chi_V(g) = \sum_i w_k.$$

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