Swag lecture 9

Wednesday, October 28, 2020 10:03 AM

Last time

Thm Lee V be a rep of G, $g \in G$.

(a) $\chi_V(e) = \dim V$

(b) $\chi_V(g)$ is a sum of roots of unity

(c) $\chi_V(g^{-1}) = \overline{\chi_V(g)}$

(d) $\chi_{V \oplus W} = \chi_V + \chi_W$

(e) $\overline{\chi_V}$ is also a character

$\overline{\chi_V}(g) = \overline{\chi_V(g)}$

Pf:

(b) Let $H = \langle g \rangle$ be the subgroup generated by g, and restrict $\rho_V|_H \Rightarrow$ a rep of H.

H is an abelian, cyclic group,

consider V as a rep of H, then decompose

$V = V_1 \oplus \cdots \oplus V_k$ V_i irr

since every irr rep of abelian group is 1-dim

$\Rightarrow V_i$ is one dimension.

if $v_i \in V_i$, then

$\rho_V(g)(v_i) = w_i \cdot v_i$,

$w_i^{|H|} = 1$

moreover, $(v_1, \cdots, v_k)$ is a basis of V,

wrt to this basis,

$$\rho(g) = \begin{pmatrix} w_1 & & 0 \\ & w_2 & \\ 0 & & \ddots \; w_k \end{pmatrix}$$

$\chi_V(g) = \sum_i w_k$. ∎

(c). by (b), $\rho(g^{-1}) = \begin{pmatrix} w_1^{-1} & & \\ & \ddots & \\ & & w_k^{-1} \end{pmatrix}$

So $\chi_V(g^{-1}) = \sum_{i=1}^{k} w_i^{-1} = \sum_{i=1}^{k} \overline{w_i} = \overline{\chi_V(g)}$

(w_i are roots of unity)

(d): $\chi_{V \oplus W} = \chi_V + \chi_W$.

find a basis for $V \oplus W$, s.t.

$\rho_{V \oplus W}(g) = \begin{pmatrix} \rho_V(g) & 0 \\ 0 & \rho_W(g) \end{pmatrix}$ for $\forall g \in G$.

So $Tr(\rho_{V \oplus W}(g)) = Tr(\rho_V(g)) + Tr(\rho_W(g))$

(e) $\overline{\chi_V}$: $\overline{\chi_V}(g) = \overline{\chi_V(g)}$

is a character $\Rightarrow$ find a rep ρ^* s.t. $\overline{\chi_V}$ is the character of ρ^*.

by some exercise, ρ^* is given by

$\rho^*(g) = \rho(g^{-1})^T$ one can check ρ^* is a rep

$\Rightarrow \chi_{\rho^*}(g) = Tr(\rho(g^{-1})^T) = Tr(\rho(g^{-1}))$
$= \chi_V(g^{-1})$
$= \overline{\chi_V(g)}$. ∎

Def:

(a) following (a) in Last Thm,

$\chi(1)$ is called the degree of χ.

(1: id in G)

(b) Let χ, ψ be 2 characters of G,

then $\langle \chi, \psi \rangle := \frac{1}{|G|} \sum_{g \in G} \overline{\chi(g)} \psi(g)$.

(1) $\langle \chi, \psi \rangle = \overline{\langle \psi, \chi \rangle}$

(2) $\langle \chi + \chi', \psi \rangle = \langle \chi, \psi \rangle + \langle \chi', \psi \rangle$

$\langle \chi, \psi + \psi' \rangle = \langle \chi, \psi \rangle + \langle \chi, \psi' \rangle$

(3) $\langle \chi, \chi \rangle \geq 0$ it is 0 iff $\chi = 0$.

Thm (orthogonality of characters)

Let V, W be irr rep, then,

(a) $\langle \chi_V, \chi_V \rangle = 1$

(b) if $V \ncong W$, then $\langle \chi_V, \chi_W \rangle = 0$

Pf: will be proved in lecture 10.

Cor Let V be a rep of G, and I is an irr rep of G, then the number of times I appears in a decomposition of V into irr reps is $\langle \chi_V, \chi_I \rangle$.

Pf: $V = V_1 \oplus V_2 \cdots \oplus V_k$, V_i irr,

$\chi_V = \chi_{V_1} + \chi_{V_2} \cdots + \chi_{V_k}$,

$\langle \chi_V, \chi_I \rangle = \langle \chi_{V_1} + \cdots + \chi_{V_k}, \chi_I \rangle$
$= \sum_i \langle \chi_{V_i}, \chi_I \rangle$
$= \sum_i \delta_{\{V_i \cong I\}}$

RMK: this number is independent of decomposition, because χ_V and χ_I are.

Cor: Let V be a rep of G,

V is irr iff $\langle \chi_V, \chi_V \rangle = 1$

Pf $V \cong I_1^{n_1} \oplus \cdots \oplus I_l^{n_l}$ ($I_i \ncong I_j$)

n_i: non negative.

then $\langle \chi_V, \chi_V \rangle = \sum_{1 \leq j,k \leq l} \langle \chi_{I_j}, \chi_{I_k} \rangle n_j n_k$.

$= \sum_{j=1}^{l} \langle \chi_{I_j}, \chi_{I_j} \rangle n_j^2$.

$= \sum_{j=1}^{l} n_j^2$

So if $\langle \chi_V, \chi_V \rangle = 1 \Leftrightarrow \sum_j n_j^2 = 1$

$\Leftrightarrow l = 1, n_j = 1$

$V \cong I_1$ ∎

Cor Let $\{I_1 \cdots I_k\}$ be a complete set of irr rep of G, then

$\mathbb{C}[G] \cong I_1^{\dim I_1} \oplus \cdots \oplus I_k^{\dim I_k}$.

Pf: $\mathbb{C}[G] = I_1^{n_1} \oplus \cdots \oplus I_k^{n_k}$.

$n_i = ?$

$n_j = \langle \chi_{\mathbb{C}[G]}, \chi_{I_j} \rangle$

$n_j = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_{reg}(g)} \chi_{I_j}(g)$

Recall: The character of the regular rep ρ_{reg} is given by

$$\rho_{reg}(g) = \begin{cases} |G| & g = 1 \\ 0 & g \neq 1 \end{cases}$$

$n_j = \frac{1}{|G|} \cdot \overline{\chi_{reg}(1)} \cdot \chi_{I_j}(1)$

($\chi_{reg}(1) = |G|$, $\chi_{I_j}(1) = \dim(I_j)$)

$\Rightarrow n_j = \dim(I_j)$.

Cor. $|G| = \sum_{j=1}^{k} \dim(I_j)^2$.

Pf: $\mathbb{C}[G] \cong I_1^{\dim I_1} \oplus \cdots \oplus I_k^{\dim I_k}$

$\dim(\mathbb{C}[G]) = |G|$.

∎