The tableaux algebra, with applications to geometry and crystals.

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§1. Tableaux algebra.

§2. Combinatorics

§3. Geometry.

§4. Crystals.

1. Tableaux algebra.

I probably don't need to justify why semistandard Young tableaux are extremely interesting objects with many applications. In this talk I will mostly touch on their combinatorial and geometric implications.

Throughout $\mathbb{k}$ is an algebraically closed field of characteristic zero. This is not necessary for all the results, but let's leave it here for safety.

Denote by $SSYT_m^n$ the set of semistandard Young tableaux of any shape with at most n rows and entries in $\{1,\dots,m\}$, so in particular $m \geq n$.

Definition: Given T with shape λ and T' with shape μ, define $T * T'$ to be the semistandard Young tableaux of shape $\lambda+\mu$ obtained by horizontally concatenating the rows and then sorting the entries in each row in weakly increasing order from left to right.

Example:

$$\begin{ytableau} 1 & 1 \\ 2 & 3 \\ 4 \end{ytableau} * \begin{ytableau} 1 & 2 \\ 2 \\ 3 \end{ytableau} = \begin{ytableau} 1 & 1 & 1 & 2 \\ 2 & 2 & 3 \\ 3 & 4 \end{ytableau}$$

Proposition: The triple $(SSYT^n_m, *, \emptyset)$ is a cancellative, commutative, reduced, torsion-free, multigraded monoid, generated by the set of columns:

$${}_nG_m := \{ T \in SSYT^n_m(1^k) \mid 1 \leq k \leq n \}.$$

cancellative: $ab = ac \Rightarrow b = c$.
reduced: $\emptyset$ is the only unit.
torsion-free: $a^n = b^n \Rightarrow a = b$.

Definition: The <u>tableaux algebra</u> ${}_nT_m$ is the monoid algebra of $SSYT^n_m$.

Theorem: The tableaux algebra ${}_nT_m$ is:

(1) Finitely generated over $\mathbb{k}$ with ${}_nG_m$ as a minimal generating set.

(2) An integral domain (so reduced, in particular).

(3) Noetherian.

(4) Jacobson.

(5) with zero Jacobson radical: $J({}_nT_m) = \bigcap_{M \in \mathrm{maxSpec}({}_nT_m)} M = 0.$

reduced: no non-zero nilpotents.

Jacobson: every prime ideal is an intersection of maximal ideals.

Any finitely generated ring over a Jacobson ring is Jacobson.

${}_nT_m$ is finitely generated over $\mathbb{k}$, and $\mathbb{k}$ is Jacobson.

$J({}_nT_m) = 0$:

Any commutative and finitely generated algebra over a field has Jacobson radical equal to its nilradical. The nilradical of ${}_nT_m$ is zero because it is reduced because it is an integral domain.

In particular, ${}_nT_m$ is semiprimitive because $J({}_nT_m) = 0$. However, ${}_nT_m$ is not primitive and not Artinian because it is a commutative integral domain that is not a field.

semiprimitive: there is a faithful semisimple module M; ${}_nT_m \hookrightarrow \mathrm{End}(M)$.

$\bigoplus_{M \in \mathrm{maxSpec}({}_nT_m)} {}_nT_m/M$ is a faithful semisimple module.

primitive: there is a faithful simple module.

Example:

$$
{}_2T_3 = \frac{\mathbb{k}\left[\begin{array}{|c|}\hline 1\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline\end{array}, \begin{array}{|c|}\hline 3\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\right]}{\left(\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array} \cdot \begin{array}{|c|}\hline 1\\\hline\end{array} - \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} \cdot \begin{array}{|c|}\hline 2\\\hline\end{array}\right)}
$$

Theorem: The tableaux algebra ${}_nT_m$ is Koszul.

<u>Sketch of the proof:</u> Clearly uT_m is quadratic, setting $uV_m := \mathrm{Span}_{\mathbb{k}}\{uG_m\}$ then

$$uT_m = \frac{T(uV_m)}{\langle R \rangle}$$ where the relations are generated by:

$$T_1 \otimes T_2 - T_2 \otimes T_1 \qquad \text{(commutation)}$$

$$T_1 \otimes T_2 - T_3 \otimes T_4 \qquad \text{(product)}$$

where $T_i \in uG_m$. We also have a total order $T < T'$ when:

(1) T has less columns than T', or

(2) T and T' have the same number of columns and the column reading word of T is smaller than the one of T'.

We can now present the relations above in terms of generators satisfying:

$$(T_1, T_2) > (T_2, T_1), \quad (T_3, T_4) > (T_4, T_3), \text{ and } (T_1, T_2) > (T_3, T_4).$$

This makes uT_m a PBW algebra, so Koszul. $\square$.

[Quadratic algebras by Polishchuck and Positselski.]

<u>2. Combinatorics</u>

Let uP_m be the ideal in $\mathbb{k}[uG_m]$ generated by product relations. Let:

$$ {}_nF_n := \left\{ \begin{array}{|c|}\hline n \\ \hline \end{array}, \begin{array}{|c|}\hline 1 \\ \hline \vdots \\ \hline n-1 \\ \hline \end{array}, \begin{array}{|c|}\hline 1 \\ \hline \vdots \\ \hline n-1 \\ \hline n \\ \hline \end{array} \right\}, \quad {}_nF_m := \left\{ \begin{array}{|c|}\hline m \\ \hline \end{array}, \begin{array}{|c|}\hline 1 \\ \hline \vdots \\ \hline n \\ \hline \end{array} \right\} \text{ with } m>n, \text{ and} $$

$$ {}_nE_m := {}_nG_m \setminus {}_nF_m . $$

Then ${}_nT_m \cong \dfrac{\mathbb{k}[{}_nE_m]}{{}_nP_m} \otimes \mathbb{k}[{}_nF_m]$. Namely, $T \in {}_nG_m$ satisfies a product relation if and only if $T \in {}_nE_m$, which is proven using an inductive argument. This distinction between columns satisfying a product relation and columns that do <u>not</u> satisfy a product relation is essential.

<u>Question:</u> What is the size of a minimal generating set of ${}_nP_m$? The combinatorialists in the project were baffled that we could <u>not</u> find our computations in OEIS.

The naive idea was to make some computer experiments, make a guess for the size, and prove the size. Proving it was a bit more tricky.

<u>Idea:</u> A product relation is essentially coming from a semistandard Young tableau with two columns that admit "swaps" in the rows:

$$ {}_2G_3 \ni \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline \end{array}, \begin{array}{|c|}\hline 2 \\ \hline \end{array} \leftsquigarrow \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} \rightsquigarrow \begin{array}{|c|}\hline 1 \\ \hline \end{array}, \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline \end{array} \in {}_2G_3 . $$

Setup: Given $T, T' \in {}_nG_m$, let $L_{T,T'}$ be the left column of $T * T'$ and let $R_{T,T'}$ be the right column of $T * T'$. It turns out that given two pairs of columns (S,T) and (S',T') with $S * T = S' * T'$, we can order them by declaring $(S,T) \triangleleft (S',T')$ when S is taller than S', and if S and S' have the same height, when $T < T'$. Observe that there is <u>no</u> need to compare T and T' because if the heights of S and S' are equal and $S = S'$ then automatically $T = T'$ because $S * T = S' * T'$. Moreover, among the pairs of columns (T, T') with $T * T' = S$ fixed, the minimal element is well defined and is $(L_{T,T'}, R_{T,T'})$. We can then count pairs of pairs $((L_{T,T'}, R_{T,T'}), (T,T'))$ satisfying $(L_{T,T'}, R_{T,T'}) \triangleleft (T,T')$ and $(L_{T,T'}, R_{T,T'}) \neq (T,T')$. This turns out to boil down to a classical weak ballot counting problem!

[Memoir on the theory of numbers IV by P.A. MacMahon.]

3. Geometry.

The tableaux algebra has a very nice geometric interpretation.

Theorem: The zero locus $\mathcal{V}(\mathfrak{n}\mathfrak{P}_n)$ is a toric variety.

Proof: A prime binomial ideal gives a toric variety by work of Eisenbud and Sturmfels.

The ideal $\mathfrak{n}\mathfrak{P}_n$ is prime because $\mathfrak{n}\mathfrak{T}_n$ is an integral domain, and it is binomial because the generators are the product relations.

After spending five years at A&M hearing Laura Matusevich and Frank Sottile talk about toric varieties, you wonder if you should have paid more attention... The answer is yes!

Here goes a motivating sentence, no definitions. Note that $SSYT_n^n$ is in bijection with GT_n the set of Gelfand-Tsetlin patterns, a bijection that induces an isomorphism between the tableaux algebra $\mathfrak{n}\mathfrak{T}_n$ and the Gelfand-Tsetling semigroup ring $\mathbb{C}T_n$. Among many remarkable properties, $\mathbb{C}T_n$ is related to the Plücker algebra $\mathbb{k}[p_\sigma \mid \sigma \subseteq \{1,\dots,n\}]/(\text{Plücker relations})$, which is essentially given by all the minors of an invertible $n\times n$ matrix of formal variables. When quotienting by the initial ideal of the ideal of Plücker relations we obtain:

$$\mathfrak{n}\mathfrak{T}_n \cong \mathbb{C}T_n \cong \text{Plücker algebra}\Big/\left(\begin{array}{c}\text{initial ideal of the ideal}\\ \text{of Plücker relations}\end{array}\right)$$

where the right hand side is a one-parameter flat degeneration of the Plücker algebra.

> Given $f: X \to Y$ a morphism of schemes, the collection $\{f^{-1}(y) \mid y \in Y\}$ is a flat family if the structure sheaf $\mathcal{O}_X$ is flat over Y, namely $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{Y,f(x)}$-module for all $x \in X$.

Intuitively, a flat degeneration resembles a formal deformation: for an algebra A, which induces a multiplication on $A[t]$ such that quotienting by (t) gives A_0, we say that A_0 is a "flat degeneration" of $A[t]$.

<u>Theorem</u>: The algebra ${}_nT_n$ is a flat degeneration of the Plücker algebra. The zero locus $V({}_n\mathfrak{B}_n)$ is a toric degeneration of the complete flag variety.

This relies on work of Kogan and Miller.

The complete flag variety is the variety associated to the Plücker algebra.

With a bit of mix-and-match we also get the following.

<u>Theorem</u>: The algebra ${}_nT_m$ is a flat degeneration of $\bigoplus_{\underline{a}} \Gamma(SL_m/{}_nQ_m, L^{\underline{a}})$. The zero locus $V({}_n\mathfrak{B}_m)$ is a toric degeneration of the partial flag variety

$SL_m/_{n}Q_m$.

Here $L^{\hat{}}$ is the tensor product of the ample generators of $\mathrm{Pic}(SL_m/P_K)$.

This relies on work of Gonciulea and Lakshmibai.

<u>Corollary:</u> The minimal generating set ${}_n\mathcal{B}_m := \{T\cdot T' - L_{T,T'}\cdot R_{T,T'}\}$ counted above is a Gröbner basis of ${}_n\mathcal{P}_m$.

All these proofs rely heavily on lattices and the work of Hibi.

<u>Observation:</u> Something fishy is happening; the complete flag variety:

$$Fl(m,m) := \{0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_m = \mathbb{K}^m \mid \dim(V_i) = i\}$$

and the partial flag variety:

$$Fl(n,m) := \{0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n \subsetneq \mathbb{K}^m \mid \dim(V_i) = i\}$$

coincide for $n = m-1$:

$$Fl(m-1,m) = \{0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{m-1} \subsetneq \mathbb{K}^m \mid \dim(V_i) = i\} = Fl(m,m).$$

Also ${}_mT_m$ is a flat degeneration of $Fl(m,m)$ and ${}_{m-1}T_m$ is a flat degeneration of $Fl(m-1,m)$, namely the same geometric object. However, ${}_mT_m \neq {}_{m-1}T_m$ because ${}_mT_m$ has one extra polynomial variable, although $\mathcal{V}({}_m\mathcal{P}_m) \cong \mathcal{V}({}_{m-1}\mathcal{P}_m)$ seems to hold. What is going on? Well, this last equivalence is slightly misleading because the ambient space is different! The explanation is that we are seeing the flag varieties within products of projective spaces in two distinct ways.

$$\begin{array}{ccccc} Fl(m,m) & \hookrightarrow & \prod_{i=1}^{m} Gr(i,m) & \xhookrightarrow{\prod_{i=1}^{m} \iota_i} & \prod_{i=1}^{m} \mathbb{P}^{\binom{m}{i}-1} = \prod_{i=1}^{m-1} \mathbb{P}^{\binom{m}{i}-1} \times \mathbb{P}^0 \\ (V_1 \subsetneq \cdots \subsetneq V_m) & \longmapsto & (V_i)_{i=1}^{m} & & \end{array}$$

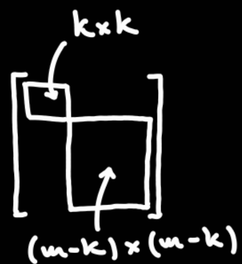
$$\begin{array}{ccccc} Fl(m-1,m) & \hookrightarrow & \prod_{i=1}^{m-1} Gr(i,m) & \xhookrightarrow{\prod_{i=1}^{m-1} \iota_i} & \prod_{i=1}^{m-1} \mathbb{P}^{\binom{m}{i}-1} \\ (V_1 \subsetneq \cdots \subsetneq V_{m-1}) & \longmapsto & (V_i)_{i=1}^{m-1} & & \end{array}$$

The extra point in the ambient space of $Fl(m,m)$ corresponds to the additional polynomial variable of $_mT_m$.

<u>Remark:</u> This discrepancy can also be noticed and explained in the construction of

$\bigoplus_{\lambda} \Gamma(SL_m/_nQ_m, L^{\lambda})$. Here we are taking for $m>n$:

$$_nQ_m = \bigcap_{k=1}^{n} P_k \quad \text{with } P_k \subsetneq SL_m \text{ the maximal parabolic}$$



and for $n=m$ we are taking:

the Plücker algebra, including the full $m \times m$ minor as an additional variable.

This extra variable corresponds to $k=0$, whereas P_k is <u>not</u> maximal (because it is <u>not</u> proper).