Inner workings of the Hochschild cohomology of some twisted tensor products

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Abstract

The Hochschild cohomology is a tool for studying associative algebras that has a lot of structure: it is a Gerstenhaber algebra, and computing this structure is difficult. We will give a mild introduction to this cohomology, as well as some of the recent developments by Volkov on how to understand it independently of the resolution. We will then present a result by Le and Zhou that motivates our line of work: it justifies looking at (twisted) tensor products of algebras. This object will be defined and the resolution-focused approach by Shepler and Witherspoon will be sketched. Our contributions will follow, including how to compute the Gerstenhaber bracket on twisted tensor products. These new results extend and generalize the existing literature, including Le and Zhou's result. This is joint work with Tekin Karadag, Dustin McPhate, Tolulope Oke, and Sarah Witherspoon.

Contents

1 Hochschild cohomology

Definition 1. *Let A be a k-algebra (our algebras are unital and associative, I'm not a monster. We define the Hochschild cohomology as):* $\mathrm{HH}^n(A) = \mathrm{Ext}^n_{A^e}(A, A)$ *where* $A^e = A \otimes A^{op}$ *(is called the enveloping algebra of A). It comes with two operations (defined on cochains):*

$$\smile : \mathrm{HH}^m(A) \times \mathrm{HH}^n(A) \longrightarrow \mathrm{HH}^{m+n}(A),$$
$$[-,-] : \mathrm{HH}^m(A) \times \mathrm{HH}^n(A) \longrightarrow \mathrm{HH}^{m+n-1}(A).$$

As is common when studying complicated objects, we can tackle problems by reducing them to an easier case. The cup product makes $\mathrm{HH}^*(A)$ into a graded commutative algebra, and now in the world of commutative things hopefully understanding this is easier. However, the payoff is that the bracket is quite complicated.

The two operations are called the cup product and the Gerstenhaber bracket, and together with some compatibility conditions, make $\mathrm{HH}^*(A)$ into a Gerstenhaber algebra. This structure can be thought of as a graded Lie algebra. Since this defines the $\mathrm{HH}^n(A)$ as an Ext group, it does not matter which projective resolution of A we choose when looking at this cohomology. However, both operations are natively defined on the "bar resolution".

Definition 2. *For any $n \in \mathbb{N}$, consider $A^{\otimes(n+2)}$ as an A^e-module (by multiplication on the outermost factors) and the sequence:*

$$\cdots \xrightarrow{d_3} A^{\otimes 4} \xrightarrow{d_2} A^{\otimes 3} \xrightarrow{d_1} A \otimes A \xrightarrow{\mu_A} A$$

with

$$d_n(a_0 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots a_{n+1}$$

for all $a_0, \dots, a_{n+1} \in A$. This is (called) the (augmented) bar resolution of A.

Notice that it is a complex in the same way as the simplicial complex of a triangulation of a manifold, and it is also exact, in fact, it has a contracting homotopy

$$s_n(a_0 \otimes \cdots \otimes a_{n+1}) = 1 \otimes a_0 \otimes \cdots \otimes a_{n+1}.$$

The bar resolution is a very nice one, big but somewhat reasonable to work with, and free as A^e-modules. I will spare you the full computation of the $\mathrm{HH}^*(A)$ of any algebra, but we can do some of the lower degrees together. Notice how the following computations rely heavily on inside knowledge of the resolution being used.

- Degree 0: we have $\mathrm{HH}^0(A) = \ker(d_1^*)$, so pick any $\alpha \in \hom_{A^e}(A \otimes A, A)$ also in $\ker(d_1^*)$, we have for all $a \in A$:

  $$0 = d_1^*(\alpha)(1 \otimes a \otimes 1) = \alpha(d_1(1 \otimes \alpha \otimes 1)) = \alpha(a \otimes 1 - 1 \otimes a) = a\alpha(1 \otimes 1) - \alpha(1 \otimes 1)a$$

  Notice that since α is determined by its value $\alpha(1 \otimes 1) \in A$, this is enough. Conversely, any element of the algebra A defines a function in $\hom_{A^e}(A \otimes A, A)$, and additionally for all $z \in Z(A)$ and $a, b \in A$ setting $\alpha_z(a \otimes b) = azc$ gives a function $\alpha_z \in \hom_{A^e}(A \otimes A, A)$ also in $\ker(d_1^*)$. Thus as k-modules we have that $\mathrm{HH}^0(A) \cong Z(A)$.

- Degree 1: we have $\mathrm{HH}^1(A) = \ker(d_2^*)/\operatorname{im}(d_1^*)$. Doing an analogous analysis to the above, we obtain that as k-modules $\ker(d_2^*) \cong \mathrm{Der}(A, A)$ the space of k-derivations from A to A, $\operatorname{im}(d_1^*) \cong \mathrm{InnDer}(A, A)$ the space of inner k-derivations from A to A. Hence as a k-module $\mathrm{HH}^1(A) \cong \mathrm{OutDer}(A, A)$ the space of outer k-derivations from A to A.

- Degree 2: we have $\mathrm{HH}^2(A) = \ker(d_3^*)/\operatorname{im}(d_2^*)$. Doing a similar analysis as done above, we obtain that as k-modules $\ker(d_3^*)$ are the infinitesimal deformations of A, and $\operatorname{im}(d_2^*)$ are the infinitesimal deformations of A that give an algebra isomorphic to the original A. Hence we can think of $\mathrm{HH}^2(A)$ as encoding the "important" infinitesimal deformations. I will not define what an infinitesimal deformation is, but the name is quite suggestive.

This hints at $\mathrm{HH}^*(A)$ encoding algebraically some form of infinitesimal information of A. Hopefully this justifies the interest and usefulness of Hochschild cohomology in representation theory, homological algebra, and other areas.

Definition 3. *(Given A a k-algebra,) let $\mu_P : P \to A$ be a resolution of A-bimodules, $\Delta_P : P \to P \otimes_A P$ a diagonal map, and $\alpha \in \mathrm{Hom}_{A^e}(P_m, A)$ a cocycle. A homotopy lifting (of α with respect to Δ_P) is (an A-bimodule chain homomorphism) $\psi_\alpha : P \to P[1-m]$ satisfying (some* *very) technical conditions (depending only on the augmentation map μ_P, the diagonal map Δ_P, and the cocycle α:*

$$d(\psi_\alpha) = (\alpha \otimes 1_P - 1_P \otimes \alpha)\Delta_P, \quad \text{and} \quad \mu_P\psi_\alpha \text{ is cohomologous to } (-1)^{m-1}\alpha\psi$$

for some A-bimodule chain map $\psi : P \to P[1]$ for which $d(\psi) = (\mu_P \otimes 1_P - 1_P \otimes \mu_P)\Delta_P$).

The technical conditions do not require inside knowledge of the resolution, which is of vital importance. Moreover, Volkov proved that for any resolution, for any diagonal, and for any cocycle, homotopy liftings always exist! Moreover, they induce the Gerstenhaber bracket in cohomology! This is absolutely fantastic.

Theorem 4. *The bracket (given at the chain level by):*

$$[\alpha, \beta] = \alpha\psi_\beta - (-1)^{(m-1)(n-1)}\beta\psi_\alpha$$

induces the Gerstenhaber bracket (on Hochschild cohomology).

2 Twisted tensor product of algebras

The motivating result for looking at twisted tensor products is the following result.

Theorem 5 (Le-Zhou 2014)**.** *Let A and B be k-algebras, at least one of them finite dimensional. Then (as Gerstenhaber algebras):*

$$\mathrm{HH}^*(A \otimes B) \cong \mathrm{HH}^*(A) \otimes \mathrm{HH}^*(B).$$

This was proven using the cumbersome Alexander-Whitney and Eilenberg-Zilber maps.

Definition 6. *Let A and B be k-algebras, a twisting map $\tau : B \otimes A \to A \otimes B$ is a bijective k-linear map (with the conditions $\tau(1_B \otimes a) = a \otimes 1_B$, $\tau(b \otimes 1_A) = 1_A \otimes b$ for all $a \in A$, $b \in B$, and:*

$$\tau \circ (m_B \otimes m_A) = (m_A \otimes m_B) \circ (1 \otimes \tau \otimes 1) \circ (\tau \otimes \tau) \circ (1 \otimes \tau \otimes 1)$$

or equivalently

$$\begin{tikzcd}
B \otimes B \otimes A \otimes A \arrow[r, "1\otimes\tau\otimes 1"] \arrow[d, "m_B\otimes m_A"'] & B \otimes A \otimes B \otimes A \arrow[r, "\tau\otimes\tau"] & A \otimes B \otimes A \otimes B \arrow[d, "1\otimes\tau\otimes 1"] \\
B \otimes A \arrow[r, "\tau"] & A \otimes B & A \otimes A \otimes B \otimes B \arrow[l, "m_A\otimes m_B"']
\end{tikzcd}$$

is a commutative diagram). The twisted tensor algebra $A \otimes_\tau B$ is $A \otimes B$ (as a vector space) with (as it turns out associative) multiplication:

$$m_{A\otimes_\tau B} : A \otimes B \otimes A \otimes B \xrightarrow{1\otimes\tau\otimes 1} A \otimes A \otimes B \otimes B \xrightarrow{m_A\otimes m_B} A \otimes B.$$

Example 7. *Let A, B be k-algebras graded by the commutative groups F, G respectively, let $t : F \otimes_{\mathbb{Z}} G \to k^\times$ be a bicharacter. Then $\tau(b \otimes a) = t(|a|, |b|)a \otimes b$ is a twisting map, we denote $A \otimes^t B = A \otimes_\tau B$.*

Theorem 8 (Shepler-Witherspoon 2019)**.** *Under some compatibility conditions, given $P \to A$, $Q \to B$ (projective bimodule resolutions of A, B respectively, and a twisting map $\tau : B \otimes A \to A \otimes B$), we can construct $P \otimes_\tau Q \to A \otimes_\tau B$ (a projective bimodule resolution of $A \otimes_\tau B$).*

3 Our contributions and applications

Theorem 9 (KMOOW)**.** *Let $P \to A$, $Q \to B$ (projective bimodule resolutions of A, B respectively) such that $P \otimes_\tau Q \to A \otimes_\tau B$ is nice (a counital differential graded coalgebra) and $\sigma : (P \otimes_\tau Q) \otimes_{A\otimes_\tau B} (P \otimes_\tau Q) \to (P \otimes_A P) \otimes_\tau (Q \otimes_B Q)$ is (also) nice (a chain map isomorphism satisfying some technical conditions). Then the (Gerstenhaber) bracket (is given) explicitly.*

The condition over σ is that:

$$\mu_P \otimes \mu_Q \otimes 1_P \otimes 1_Q - 1_P \otimes 1_Q \otimes \mu_P \otimes \mu_Q = (\mu_P \otimes 1_P \otimes \mu_Q \otimes 1_Q - 1_P \otimes \mu_P \otimes 1_Q \otimes \mu_Q)\sigma$$

and the bracket is given by $[\alpha, \beta] = \alpha\psi_\beta - (-1)^{(m-1)(n-1)}\beta\psi_\alpha$ with $\psi_\alpha = \phi(1 \otimes \alpha \otimes 1)\Delta^{(2)}$ where $\phi = (\phi_P \otimes \mu_Q \otimes 1_Q + 1_P \otimes \mu_P \otimes \phi_Q)\sigma$ where ϕ_P is a contracting homotopy for $\mu_P \otimes 1_P - 1_P \otimes \mu_P$.

This allows computing the Gerstenhaber bracket in the Hochschild cohomology of a twisted tensor product $A \otimes_\tau B$, a notoriously difficult task, as long as we know the Gerstenhaber bracket in the respective Hochschild cohomologies of A and B. This has applications in, for example, deformations of algebras.

Proof. (The proof of this method uses fairly) elementary methods (besides the homotopy lifting techniques, which are borrowed from a paper by Negron and Witherspoon and essentially remain a black box. All the compatibility conditions can be translated into commutative diagrams, and by filling them up we are essentially done). □

It can be checked that if A and B are graded by the commutative groups F and G respectively, then $\mathrm{HH}^*(-)$ is bigraded: $\mathrm{HH}^{*,*}(-)$. In the context of the twisting by a bicharacter, we also denote:

$$F' = \bigcap_{g\in G} \ker(t(-, g)), \quad G' = \bigcap_{f\in F} \ker(t(f, -)).$$

Theorem 10 (Grimley-Nguyen-Witherspoon 2017, OOW)**.** *As Gerstenhaber algebras (in the twisted tensor product setup, and assuming the necessary finiteness conditions, we have):*

$$\mathrm{HH}^{*,F'\oplus G'}(A \otimes^t B) \cong \mathrm{HH}^{*,F'}(A) \otimes \mathrm{HH}^{*,G'}(B).$$

Proof. (The original proof used extended versions of the Alexander-Whitney and Eilenberg-Zilber maps. We completely avoided them by using) Volkov's homotopy lifting (techniques, as well as a chain isomorphism similar to the aforementioned σ, and a bit of work with the Koszul sign convention). □

4 Remarks and future work

1. We did not use the explicit formula for σ (at least not its full expression).

   The original proofs required the explicit expression of σ because of the use of the Alexander-Whitney and Eilenberg-Zilber maps, but we only used that σ makes some diagrams commute. This should also hold for the version in [KMOOW], and is current work in progress.

2. Compute more examples.

   New examples and complete computations are always useful, the current examples are relatively small and relatively scarce.

3. Understand why some examples (like the Jordan plane) work: $k\langle x, y\rangle/(yx - xy - x^2)$.

   The complete Gerstenhaber algebra structure of the Jordan plane was first computed by Lopes and Solotar, using spectral sequences and a lot of machinery. In [KMOOW] we also computed it using more elementary and completely different methods; and although the hypothesis that we required on the twisting map were not satisfied, using these elementary techniques the conclusions of our main results held. That is, applying our constructions, we were still able to compute the complete Gerstenhaber algebra structure. What are then the correct hypothesis on the twist?

Thank you for your time!

References

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