On the Gerstenhaber bracket in relative Hochschild cohomology (of associative algebras). (1)

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Contents:

Motivation (2)

To study the cohomology of Lie algebras: [Relative Homological Algebra; Hochschild].

To understand deformations of algebras: [Gerstenhaber, Schack].

"... we should note that ... both $\smile$, $[-,-]$ induce products on relative Hochschild cohomology..."

$\boxed{\text{Theorem [0]: The relative Hochschild cohomology is a Gerstenhaber algebra.}}$

Motivation: (3)

For A an algebra over k a field: $HH^*(A) := \mathrm{Ext}^*_{A^e}(A,A)$.

For A an algebra over k commutative ring: $HH^*(A) := \boxed{\mathrm{Ext}^*_{(A^e,k^e)}(A,A)}$.

Note: $k \subseteq A$ and $k^e \subseteq A^e$.

What happens if we consider $B \subseteq A$ subalgebra?

Hochschild cohomology.

④

<u>Definition:</u> Let A be a k-algebra: $A^e := A \otimes A^{op}$.

$$\boxed{\begin{aligned} HH^n(A) &:= \mathrm{Ext}^n_{A^e}(A,A). \\ HH^*(A) &:= \bigoplus_{n \in \mathbb{N}} \mathrm{Ext}^n_{A^e}(A,A). \end{aligned}}$$

<u>Cup product:</u> $\cup : HH^m(A) \times HH^n(A) \longrightarrow HH^{m+n}(A)$

<u>Gerstenhaber bracket:</u> $[-,-] : HH^m(A) \times HH^n(A) \longrightarrow HH^{m+n-1}(A)$.

Relative homological algebra.

⑤

<u>Definition:</u> Let A be a unital associative ring, $1_A \in B \subseteq A$ a subring. Let:
$\cdots \xrightarrow{d_{i+1}} M_i \xrightarrow{d_i} M_{i-1} \xrightarrow{d_{i-1}} \cdots$ be a sequence of A-modules, it is called (A,B)-exact if it is split exact as a sequence of B-modules.

<u>Equivalently:</u>

exact as A-modules and $\mathrm{Ker}(d_i)$ direct summand of M_i.

$d_i d_{i+1} = 0$ and $1_{M_\cdot}$ is null homotopic.

Relative homological algebra.

⑥

<u>Definition:</u> An A-module P is said to be (A,B)-projective when we can complete the diagram.

(A,B)-exact row $\rightsquigarrow$

$$\begin{tikzcd} & P \arrow[dl, dashed, "h'_A"'] \arrow[d, "h_A"] & \\ M \arrow[r, "f_A"] & N \arrow[r] \arrow[l, bend left, "s_B"] & 0 \end{tikzcd}$$

$$\begin{tikzcd} & P \arrow[dl, dashed, "h'_A"'] \arrow[dl, bend left, "h_B"] \arrow[d, "g_A"] & \\ M \arrow[r, "f_A"'] & N \arrow[r] & 0 \end{tikzcd}$$

Relative homological algebra.

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This setup immediately gives notions of:

(A,B)-projective resolutions $\rightsquigarrow$ Relative comparison Theorem $\rightsquigarrow$ Relative Ext groups $\boxed{\mathrm{Ext}^n_{(A,B)}(M,N)}$

But the proofs need to be redone!

Relative homological algebra.

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It is not enough to add "(A,B)-..." everywhere, there are subtleties.

Remark:
1. A projective P is (A,B)-projective.
2. A projective resolution need not be (A,B)-projective resolution.

Relative homological algebra.

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Relative abelian categories $\leftsquigarrow$ we can use here some general constructions

lose desirable information $\rightsquigarrow$

$\leftsquigarrow$ flexibility of viewpoints.

Proofs need redoing $\rightsquigarrow$ (A,B) relative setup

Relative Hochschild cohomology: (10)

Definition: Let A be a k-algebra, $B \subseteq A$ subalgebra:

$$\boxed{\begin{aligned} HH^n(A|B, A) &:= \mathrm{Ext}^n_{(A\otimes A^{op},\, B\otimes A^{op})}(A,A) \\ HH^*(A|B, A) &:= \bigoplus_{n\in\mathbb{N}} \mathrm{Ext}^n_{(A\otimes A^{op},\, B\otimes A^{op})}(A,A). \end{aligned}}$$

Theorem [O.]: The relative Hochschild cohomology is a Gerstenhaber algebra.

Applications: (11)

[Kaygun]: Jacobi-Zariski exact sequence for Hochschild homology and cyclic (co)homology.

Under some conditions:

$$\cdots \longrightarrow HH^n(A|B, M) \longrightarrow HH^n(A, M) \longrightarrow HH^n(B, M) \longrightarrow HH^{n+1}(A|B, M) \longrightarrow \cdots$$

Applications: (12)

[Cibils, Lanzilotta, Marcos, (Schroll), Solotar]: Adding or deleting arrows of a bound quiver algebra and Hochschild (co)homology.

Describe the changes in Hochschild (co)homology when adding or deleting arrows to the quiver.

Applications:

Relaxing separability conditions required by [Gerstenhaber, Schack].

Constructions like the "relative" Yoneda product of extensions work as desired.

$\boxed{\text{Theorem [O.]: Relative Künneth Theorem.}}$

Thank you!