

On the Gerstenhaber bracket in relative Hochschild cohomology (of associative algebras). ①

Pablo S. Ocal

Texas A&M University

DG Methods in Commutative Algebra
and Representation Theory.

May 2, 2020.

Contents:

1. Motivation and result.
2. Hochschild cohomology.
3. Relative homological algebra.
4. Relative Hochschild cohomology.
5. Applications.

Motivation

To study the cohomology of Lie algebras: [Relative Homological Algebra; Hochschild].

To understand deformations of algebras: [Gerstenhaber, Schack].

"... we should note that ... both $\cup, [-, -]$ induce products
on relative Hochschild cohomology..."

Theorem [0.]: The relative Hochschild cohomology is a Gerstenhaber algebra.

Motivation:

For A an algebra over k a field: $HH^*(A) := \text{Ext}_{A^e}^*(A, A)$.

For A an algebra over k commutative ring: $HH^*(A) := \text{Ext}_{(A^e, k^e)}^*(A, A)$.

Note: $k \subseteq A$ and $k^e \subseteq A^e$.

What happens if we consider $B \subseteq A$ subalgebra?

Hochschild cohomology.

④

Definition: Let A be a k -algebra:
 $A^e := A \otimes A^{\text{op}}$

$$HH^m(A) := \text{Ext}_{A^e}^m(A, A)$$

$$HH^*(A) := \bigoplus_{n \in \mathbb{N}} \text{Ext}_{A^e}^n(A, A)$$

Cup product:

$$\cup : HH^m(A) \times HH^n(A) \rightarrow HH^{m+n}(A)$$

Gerstenhaber bracket:

$$[-, -] : HH^m(A) \times HH^n(A) \rightarrow HH^{m+n-1}(A)$$

Relative homological algebra.

⑤

Definition: Let A be a unital associative ring, $B \subseteq A$ a subring. Let:
 $\dots \xrightarrow{d_{i+1}} M_i \xrightarrow{d_i} M_{i-1} \xrightarrow{d_{i-1}} \dots$ be a sequence of A -modules, it is called (A, B) -exact if it is split exact as a sequence of B -modules.

Equivalently:

exact as A -modules
 and $\text{Ker}(d_i)$ direct
 summand of M_i .

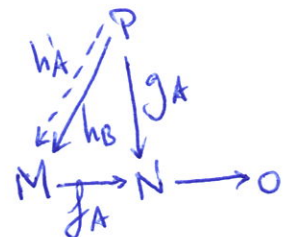
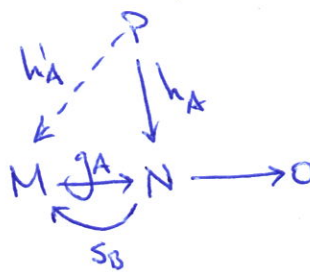
$d_i d_{i+1} = 0$ and M_i
 is null homotopic.

Relative homological algebra.

⑥

Definition: An A -module P is said to be (A, B) -projective when we can complete the diagram.

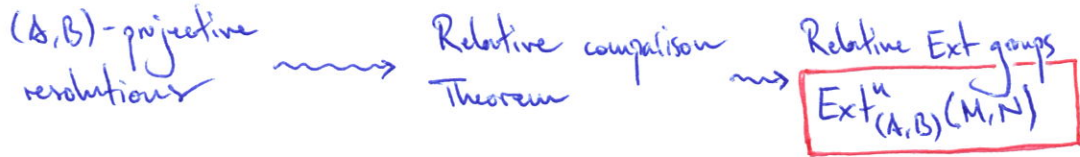
(A, B) -exact row \rightsquigarrow



Relative homological algebra.

(7)

This setup immediately gives notions of:



But the proofs need to be redone!

Relative homological algebra.

(8)

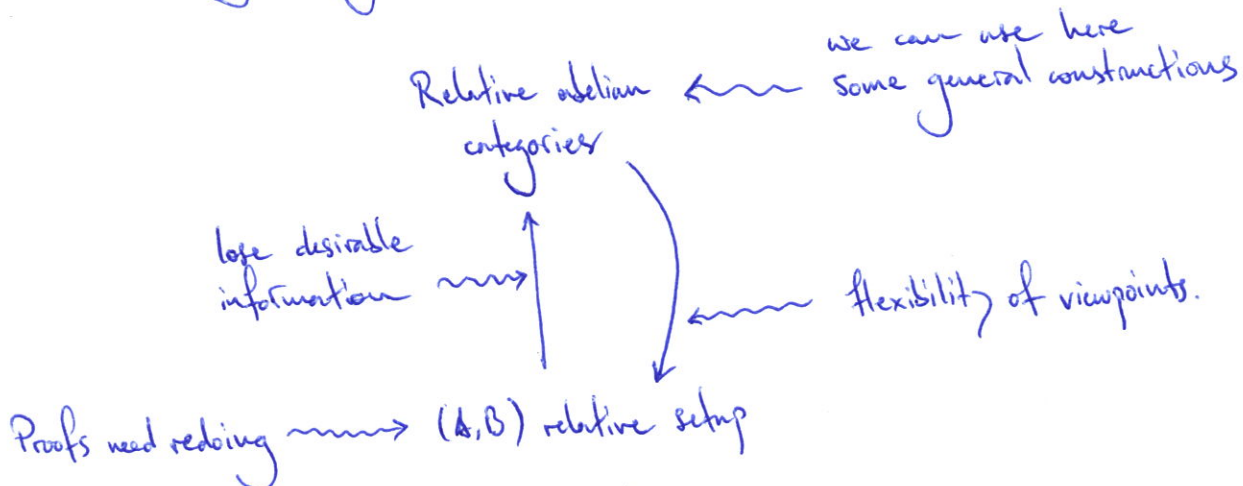
It is not enough to add " (A, B) -..." everywhere, there are subtleties.

Remark:

1. A projective P is (A, B) -projective.
2. A projective resolution need not be (A, B) -projective resolution.

Relative homological algebra.

(9)



Relative Hochschild cohomology:

(10)

Definition: Let A be a k -algebra, $B \subseteq A$ subalgebra:

$$HH^n(A/B, A) := \text{Ext}_{(A \otimes A^{\text{op}}, B \otimes A^{\text{op}})}^n(A, A)$$

$$HH^*(A/B, A) := \bigoplus_{n \in \mathbb{N}} \text{Ext}_{(A \otimes A^{\text{op}}, B \otimes A^{\text{op}})}^n(A, A).$$

Theorem [0.]: The relative Hochschild cohomology is a Gerstenhaber algebra.

Applications:

(11)

[Kazuhiko]: Jacobi-Zariski exact sequence for Hochschild homology and cyclic (co)homology.

Under some conditions:

$$\dots \rightarrow HH^n(A/B, M) \rightarrow HH^n(A, M) \rightarrow HH^n(B, M) \rightarrow HH^{n+1}(A/B, M) \rightarrow \dots$$

Applications:

(12)

[Cibils, Lanzetta, Marcos, (Schroll), Solotar]: Adding or deleting arrows of a bound quiver algebra and Hochschild (co)homology.

Describe the changes in Hochschild (co)homology when adding or deleting arrows to the quiver.

Applications:

(13)

Relaxing separability conditions required by [Gerstenhaber, Schack].

Constructions like the "relative" Yoneda product of extensions work as desired.

Theorem [0.]: Relative Künneth Theorem.

Thank you!