Quantum symmetries via twisted tensor products and their Balmer spectrum.

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Quantum symmetries.

Goal: Understand the representation theory of a Hopf algebra.

kG, $U_q(\mathfrak{g})$, $k[x]$.

Success:

→ Classify modules.

→ Classify indecomposable modules.

→ Classify simple modules.

Prototypical examples:

→ Structure theorem.

→ Highest weight theorem.

→ Finitely generated or cyclic.

Classification using categorical data.

Data: $\mathcal{C}$ category, $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ bifunctor, abelian, ...

mod(kG)

Question: When are M, N in $\mathcal{C}$ isomorphic?

kG $\mathrm{char}(k) \nmid |G|$ v.s. kG $\mathrm{char}(k) \mid |G|$

Question: When can M be built from N using categorical data?

projective modules: $P \cong \bigoplus_{i \in I} kG$

Stable module categories.

$\mathcal{C}$ Frobenius abelian category.

mod(kG)

st$(\mathcal{C})$ same objects as $\mathcal{C}$ and morphisms factoring over injectives are zero.

stmod(kG)

Triangles: M in $\mathcal{C}$, choose $M \hookrightarrow I(M) \twoheadrightarrow \Sigma(M)$ and set:

$$\begin{tikzcd} M \arrow[r, hook] \arrow[d, "f"'] & I(M) \arrow[r, two heads] \arrow[d, dashed] \arrow[dl, phantom, "\ulcorner"] & \Sigma(M) \arrow[d, equal] \\ N \arrow[r, hook, dashed] & c(f) \arrow[r, two heads, dashed] & \Sigma(M) \end{tikzcd}$$

to get $M \xrightarrow{f} N \longrightarrow c(f) \longrightarrow \Sigma(M)$.

Triangulated categories.

$K(\mathcal{A})$, $D(\mathcal{A})$, SH, motives

$\mathcal{C}$ additive, $\Sigma : \mathcal{C} \to \mathcal{C}$ autoequivalence, distinguished triangles:

$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ or
$$\begin{tikzcd} & Z \arrow[dl, "h"'] & \\ X \arrow[rr, "f"'] & & Y \arrow[ul, "g"'] \end{tikzcd}$$
satisfying:

1. $X \xrightarrow{1_X} X \to 0 \to \Sigma X$,

   $X \xrightarrow{f} Y \to \mathrm{cone}(f) \to \Sigma X$,

   closed under isomorphisms.
2. closed under rotations.
3. existence of certain morphisms.
4. octahedral axiom.

Monoidal triangulated categories.

Monoidal: $(\mathcal{C}, \otimes, \alpha, \mathbb{1}, \iota)$, pentagon axiom, unit axiom.

Tensor-triangulated: [Balmer] $(\mathcal{K}, \otimes, \mathbb{1})$ with $\mathcal{K}$ additive, triangulated, monoidal, and $\otimes : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ symmetric and exact in each variable.

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Tensor: [EGNO] $(\mathcal{T}, \otimes, \mathbb{1})$ with $\mathcal{T}$ locally finite, k-linear, abelian, rigid, monoidal, and $\otimes : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ bilinear on morphisms, and $\mathcal{T}$ indecomposable, and $\mathrm{End}_{\mathcal{C}}(\mathbb{1}) = k$.

Vec_k, $\mathrm{mod}(kG)$, $\mathrm{mod}(\mathfrak{J})$

Support data for tensor triangulated categories.

Structure: $\mathcal{C}$ **Support:** (X, σ) with $\sigma : \mathcal{C} \to \mathrm{Closed}(X)$

Additive $\begin{cases} \sigma(0) = \emptyset \\ \sigma(a \oplus b) = \sigma(a) \cup \sigma(b) \end{cases}$

Triangulated $\begin{cases} \sigma(\Sigma a) = \sigma(a) \\ \sigma(a) \subseteq \sigma(b) \cup \sigma(c) \qquad a \to b \to c \to \Sigma a \end{cases}$

Tensor $\begin{cases} \sigma(a \otimes b) = \sigma(a) \cap \sigma(b) \\ \sigma(\mathbb{1}) = X \end{cases}$

Balmer spectrum.

It is the universal (final) support data.

$$\mathrm{Spc}(\mathcal{C}) = \{ \mathcal{P} \subsetneq \mathcal{C} \mid \mathcal{P} \text{ prime thick triangulated tensor ideal of } \mathcal{C} \}$$

prime: $a \otimes b \in \mathcal{P} \Rightarrow a \in \mathcal{P}$ or $b \in \mathcal{P}$

thick: $a \in \mathcal{P},\ a \simeq b \oplus c \Rightarrow b, c \in \mathcal{P}$

triangulated: 2-out-of-3

tensor ideal: $a \in \mathcal{P},\ b \in \mathcal{C} \Rightarrow a \otimes b \in \mathcal{C}$

$\mathrm{supp}(a) = \{ \mathcal{P} \in \mathrm{Spc}(\mathcal{C}) \mid a \notin \mathcal{P} \}$ basis of closed.

Theorem [Balmer]: $\mathrm{Spc}(\mathcal{C})$ is the best space to do geometry for $\mathcal{C}$.

Comparison of spectra.

Commutative algebra:

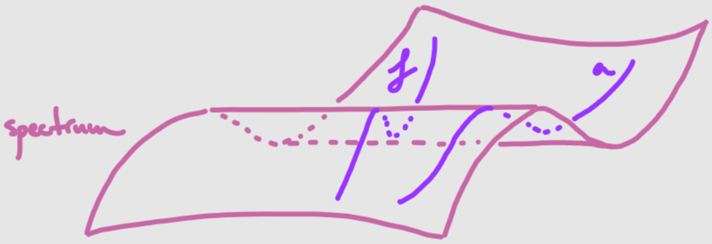
$\mathrm{Spec}(R)$ prime ideals

$f \in R$ supported at $\mathfrak{p}$ when $f \in \mathfrak{p}$

Tensor triangular geometry:

$\mathrm{Spc}(\mathcal{C})$ thick tensor ideals

$a \in \mathcal{C}$ supported at $\mathcal{P}$ when $a \notin \mathcal{P}$



f lives in $R_{\mathfrak{p}}$ when $f \in \mathfrak{p}$

a dies in $\mathcal{C}/\mathcal{P}$ when $a \in \mathcal{P}$

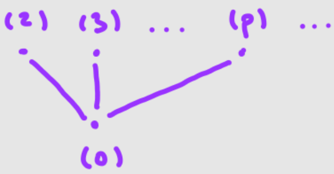
Example: reconstruction of schemes.

[Balmer] X quasi-compact quasi-separated scheme: $(\underbrace{\mathrm{Spc}(\mathcal{D}^{\mathrm{perf}}(X))}_{\text{Balmer spectrum of Koszul objects}}, \underbrace{\mathcal{O}}_{\text{sheaf of rings}}) \simeq X$.

[Neeman, Thomason] R commutative Noetherian:

$$\mathrm{Spc}(\mathcal{D}^{\mathrm{perf}}(R)) \simeq \mathrm{Spc}(K^b(\mathrm{proj}(R))) \simeq \mathrm{Spec}(R)$$

$R = \mathbb{Z}$: $\quad \mathrm{Spc}(K^b(\mathrm{proj}(\mathbb{Z}))) \simeq \mathrm{Spec}(\mathbb{Z})$



Example: bounded derived category.

[Hopkins, Neeman] $A = k[x]$:

$$\mathrm{Sp}(D^b(\mathrm{mod}(k[x]))) = \{\text{specialization closed subsets of } \mathrm{Spec}(k[x])\}.$$

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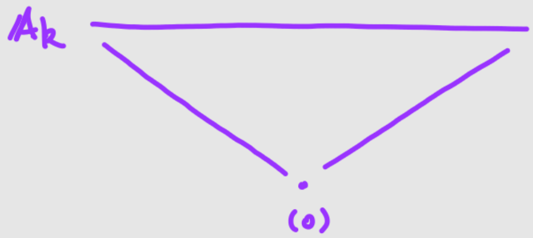
$X \subseteq \mathrm{Spec}(k[x])$ such that if $\mathfrak{p} \subseteq \mathfrak{q}$ is a pair of prime ideals of $k[x]$ with $\mathfrak{p} \in X$ then $\mathfrak{q} \in X$.

When k is algebraically closed this is the affine line:

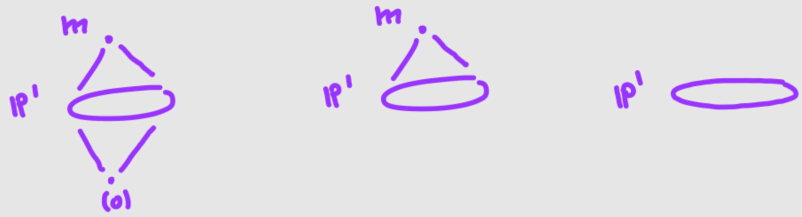


Example: representations of finite groups.

[Benson-Carlson-Rickard, Benson-Iyengar-Krause]: $\mathrm{Spc}(\mathrm{stmod}(kG)) \simeq \mathrm{Proj}(H^\bullet(G,k))$.

$G = \mathbb{Z}_2 \times \mathbb{Z}_2$: $\quad \mathrm{Spc}(\mathrm{stmod}(kG)) \simeq \mathrm{Spc}\left(\dfrac{D^b(\mathrm{mod}\, kG)}{K^b(\mathrm{proj}\, kG)}\right)$

Ideals: $\quad$ Max$(H^\bullet(G,k))$: $\quad$ Proj$(H^\bullet(G,k))$:



Twisted tensor products.

Designed to encode a non-commutative product of varieties.

$$V \times W \longrightarrow k[V] \otimes k[W]$$

$$V \times_\tau W \longrightarrow k[V] \otimes_\tau k[W]$$

Universal properties and decomposition.

Product:

$$\begin{tikzcd} A \arrow[dr, "i"] \arrow[drr, bend left, "f"] & & \\ & \Lambda \arrow[r, dashed] & C \\ B \arrow[ur, "j"] \arrow[urr, bend right, "g"'] & & \end{tikzcd}$$

$$\begin{tikzcd} B \otimes A \arrow[r, "f \otimes g"] \arrow[d, "\tau"'] & C \otimes C \arrow[dr, "\nabla_C"] & \\ & & C \\ A \otimes B \arrow[r, "g \otimes f"'] & C \otimes C \arrow[ur, "\nabla_C"'] & \end{tikzcd}$$

$$\Lambda \cong A \otimes_\tau B$$

Coproduct: Reverse all the arrows.

$$\begin{tikzcd} A & & \\ & \Lambda \arrow[ul, "p"'] \arrow[dl, "q"] & C \arrow[l, dashed] \arrow[ull, bend right, "f"'] \arrow[dll, bend left, "g"] \\ B & & \end{tikzcd}$$

$$\begin{tikzcd} B \otimes A & C \otimes C \arrow[l, "g \otimes f"'] & \\ & & C \arrow[ul, "\Delta_C"'] \arrow[dl, "\Delta_C"] \\ A \otimes B \arrow[uu, "\theta"] & C \otimes C \arrow[l, "f \otimes g"] & \end{tikzcd}$$

$$\Lambda \cong A \otimes^\theta B$$

Algebraic description.

(A, ∇_A, η_A) and (B, ∇_B, η_B) unital associative algebras.

$\tau : B \otimes A \to A \otimes B$ linear bijective preserving their structure.

Then $A \otimes_\tau B$ is a unital associative algebra.

$$\nabla_{A\otimes_\tau B} : (A\otimes B)\otimes(A\otimes B) \xrightarrow{1\otimes\tau\otimes 1} A\otimes A\otimes B\otimes B \xrightarrow{\nabla_A\otimes\nabla_B} A\otimes B,$$

$$\eta_{A\otimes_\tau B} : k \xrightarrow{\cong} k\otimes k \xrightarrow{\eta_A\otimes\eta_B} A\otimes B.$$

Examples: twisted tensor products.

Jordan plane: $A = k[x] \quad B = k[y] \quad \tau : k[y]\otimes k[x] \longrightarrow k[x]\otimes k[y]$

$$y\otimes x \longmapsto x\otimes y + x^2\otimes 1$$

$$k[x]\otimes_\tau k[y] \cong \frac{k\langle x,y\rangle}{\langle xy - yx + x^2\rangle}.$$

Quantum $\mathfrak{sl}_2$: $A = k[F] \quad B = U_{q^2}(\mathfrak{h}) \quad \tau : U_{q^2}(\mathfrak{h})\otimes k[F] \to k[F]\otimes U_{q^2}(\mathfrak{h})$

$$K\otimes F \longmapsto q^{-2} F\otimes K$$

$$E\otimes F \longmapsto F\otimes E - \frac{1\otimes K - 1\otimes K^{-1}}{q - q^{-1}}$$

$$k[F]\otimes_\tau U_{q^2}(\mathfrak{h}) \cong U_q(\mathfrak{sl}_2).$$

Cocommutative Hopf algebras over $\mathbb{C}$: $H \cong U(\mathrm{Primitive}(H)) \# \mathbb{C}\,\mathrm{Grouplike}(H)$

[Milnor, Moore, Cartier, Kostant]

Usefulness in cohomology.

Computable: $HH^\bullet(A\otimes B) \cong HH^\bullet(A)\otimes HH^\bullet(B)$ [Cartan-Eilenberg]

$HH^\bullet(A\otimes_t B) \cong HH^\bullet(A)\otimes_t HH^\bullet(B)$ [Bergh-Oppermann]

[Grimley-Nguyen-Witherspoon]

Counterexamples: $HH^\bullet(\Lambda)$ is not finitely generated. [Xu]

[Snashall-Solberg] $HH^\bullet\left(\frac{k\langle x,y\rangle}{(x^2, xy+qyx, y^2)}\right)$ is finite and $\mathrm{gldim}\left(\frac{k\langle x,y\rangle}{(x^2, xy+qyx, y^2)}\right)=\infty$.

[Happel] [Buchweitz-Green-Madsen-Solberg]

Hochschild cohomology of twisted tensor products.

Theorem [Lopes-Solotar, Karadag-McPhate-O.-Oke-Witherspoon]: Give explicit computable formulas for the Gerstenhaber algebra structure of the Jordan plane.

Theorem [Briggs-Witherspoon]: Complete description of $HH^\bullet(A\otimes_t B)$.

$A=\bigoplus_{f\in F} A_f$, $B=\bigoplus_{g\in G} B_g$, $t: F\times G \longrightarrow k^\times$ bicharacter.

Export to algebras in categories.

Theorem [EGNO]: Under some exactness and surjectivity conditions:

$$\underline{\mathcal{M}} \simeq \mathrm{Mod}_{\mathcal{C}}(A).$$

$\mathcal{M}$ is a $\mathcal{C}$-module $A = \underline{\mathrm{Hom}}(M,M)$, $M \in \mathcal{M}$

Theorem: 2D TQFTs are Frobenius algebras.

[Abrams, Kock, O.] $\mathrm{SymMonCat}(2\mathrm{Cob}, \mathcal{C}) \simeq \mathrm{cFrob}(\mathcal{C})$.

[Turaev-Turner, O.] $\mathrm{SymMonCat}(2\mathrm{UCob}, \mathcal{C}) \simeq \mathrm{c\,ExtFrob}(\mathcal{C})$.

Inheritance of Frobenius structure.

A, B Frobenius algebras.

Theorem [O.-Oswald]: $A \otimes_{\mathcal{C}} B$ is a Frobenius algebra if and only if it is a coalgebra.

Recover some quantum complete intersections: $\dfrac{k_q[x_1,\dots,x_n]}{(x_1^{m_1},\dots,x_n^{m_n})}$,

and non-commutative symmetric Frobenius algebras: $kG \otimes_{\mathcal{C}} kH$.

Spectrum of twisted tensor products.

When $A \otimes_\tau B$ is Frobenius, $\mathrm{stmod}(A \otimes_\tau B)$ is a triangulated category.

<u>Theorem [Gratz-Stevenson, Balmer-O.]</u>: Triangulated categories have universal support.

$$\mathrm{Sp}(\mathcal{C}) = \{\mathcal{I} \subseteq \mathcal{C} \mid \mathcal{I} \text{ thick triangulated subcategory of } \mathcal{C}\}$$

$\mathrm{Sp}(D^b(A_2)) \simeq$ 

Inheritance of bialgebra structure.

A, B bialgebras.

<u>Theorem [O.-Oswald]</u>: $A \otimes_\tau B$ is a bialgebra if and only if τ is trivial.

($\tau(b \otimes a) = a \otimes b$ for all $a \in A$ and $b \in B$)

$$\Delta_H : A \otimes B \xrightarrow{\Delta_A \otimes \Delta_B} A \otimes A \otimes B \otimes B \xrightarrow{1 \otimes \tau^{-1} \otimes 1} A \otimes B \otimes A \otimes B$$

$$\varepsilon_H : A \otimes B \xrightarrow{\varepsilon_A \otimes \varepsilon_B} k \otimes k \xrightarrow{\simeq} k$$

Work in progress.

Twisted and co-twisted: $H \cong A \otimes_{\tau}^{\theta} B$.

Under certain conditions: $\mathrm{Spc}(\mathrm{stmod}(H^*)) \simeq \mathrm{Proj}\left(H^{\bullet}(A, k)^{B} \right)$.

B semisimple

A not semisimple

B acting on A.

Thank you!

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