# Understanding twisted tensor products through deformations.

Pablo S. Ocal

joint with T. Karadag, D. McPhate, T. Oke, C. Valqui, S. Witherspoon.

University of California, Los Angeles

## Roadmap.

1. Introduction.
2. Twisted tensor products and deformations.
3. Hochschild cohomology.
4. Results

## 1. Introduction.

All my algebras will be unital and associative, over $k$ a field.

$A \otimes_\tau B$ : The twisted tensor product of $A$ and $B$.

Čap, Schichl, Vanžura : if an algebra has an underlying vector space given by a tensor product of two subalgebras, then it is a twisted tensor product.

$A_t$ : A formal deformation of $A$.

Gerstenhaber: an algebra can be deformed by perturbing its multiplicative structure.

$HH^*(A)$: The Hochschild cohomology of an algebra encodes infinitesimal information.

$HH^0(A) \cong Z(A)$ the center of $A$.

$HH^1(A) \cong \mathrm{OutDer}(A)$ the outer derivations of $A$.

$HH^2(A)$ the "important" infinitesimal deformations.

(the deformations giving something not isomorphic to the original algebra).

Originally the study of twisted tensor products came from non-commutative geometry, but nowadays they have applications in a variety of areas, ranging from operator algebras to algebraic topology to quantum symmetries.

## 2. Twisted tensor products and deformations.

Twisted tensor products are non-commutative generalizations of the usual tensor product.

Definition: Let $A$ and $B$ be algebras. A twisting map $\tau : B \otimes A \to A \otimes B$ is a bijective linear map such that:

1. $\tau$ preserves units:

$\tau(1_B \otimes a) = a \otimes 1_B$ and $\tau(b \otimes 1_A) = 1_A \otimes b$ for all $a \in A$, $b \in B$.

2. $\tau$ preserves multiplication (or twisting and multiplication commute):

$$\begin{tikzcd}
B \otimes B \otimes A \otimes A \arrow[r, "1 \otimes \tau \otimes 1"] \arrow[d, "m_B \otimes m_A"'] & B \otimes A \otimes B \otimes A \arrow[r, "\tau \otimes \tau"] \arrow[d, phantom, "\substack{\text{multiplying} \quad \circlearrowleft \quad \text{twisting} \\ \text{twisting} \qquad\qquad \text{multiplying}}"] & A \otimes B \otimes A \otimes B \arrow[d, "1 \otimes \tau \otimes 1"] \\
B \otimes A \arrow[r, "\tau"] & A \otimes B & A \otimes A \otimes B \otimes B \arrow[l, "m_A \otimes m_B"']
\end{tikzcd}$$

The twisted tensor algebra $A \otimes_\tau B$ has $A \otimes B$ as the underlying vector space, with multiplication:

$$m_{A \otimes_\tau B} : (A \otimes B) \otimes (A \otimes B) \xrightarrow{1 \otimes \tau \otimes 1} A \otimes A \otimes B \otimes B \xrightarrow{m_A \otimes m_B} A \otimes B.$$

This multiplication makes $A \otimes_\tau B$ into a unital associative algebra.

Deformations are a family of algebras indexed by a parameter such that their multiplications are obtained by slightly modifying a given multiplication.

Definition: Let $A$ be an algebra. A (formal) deformation $(A_t, \mu_t)$ of $A$ is $A[[t]]$ seen as a $k[[t]]$-module with an associative $k$-bilinear multiplication:

$$* : A[[t]] \otimes_{k[[t]]} A[[t]] \longrightarrow A[[t]]$$

which is fully determined by:

$$*(a \otimes b) = \mu_0(a \otimes b) + \mu_1(a \otimes b)t + \mu_2(a \otimes b)t^2 + \dots = \sum_{i \in \mathbb{N}} \mu_i(a \otimes b)t^i$$

for all $a, b \in A$, $\mu_0(a\otimes b) = m_A(a\otimes b)$, $\mu_i : A\otimes A \to A$. We extend to $A[[t]]$ via:

$$\Big(\sum_{i\in\mathbb{N}} a_i t^i\Big)\cdot\Big(\sum_{j\in\mathbb{N}} b_j t^j\Big) = \sum_{n\in\mathbb{N}}\Big(\sum_{i+j=n} a_i\cdot b_j\Big) t^n \quad \text{the Cauchy product rule.}$$

<u>Remark:</u> Twisted tensor products are a vast family of algebras, including:

(a) Non-commutative 2-tori.

(b) Crossed products of $C^*$-algebras with groups.

(c) Crossed products with Hopf algebras.

(d) Algebras with triangular decomposition (such as universal enveloping algebras of Lie algebras and quantum groups).

(e) Braided tensor product defined by R-matrices.

<u>Examples:</u> (of twisted tensor products, of iterated twisted tensor planes in fact)

1. Quantum plane or skew polynomial ring:

$$k[x_1]\otimes_\tau k[x_2] \cong \frac{k\langle x_1, x_2\rangle}{(x_1x_2 - q x_2 x_1)}, \quad \tau(x_2\otimes x_1) = q^{-1} x_1\otimes x_2, \quad q\in k^\times.$$

2. Quantum affine space:

$$k\langle x_1,\dots,x_n\rangle \big/ (x_i x_j - q_{ij} x_j x_i \mid 1\le i,j\le n) \quad \text{with } q_{ii} = 1 \text{ and } q_{ji} = q_{ij}^{-1}.$$

3. Truncated quantum plane:

$$k[x_1]/(x_1^{n_1}) \otimes_\tau k[x_2]/(x_2^{n_2}) \cong \frac{k\langle x_1, x_2\rangle}{(x_1^{n_1}, x_2^{n_2}, x_1x_2 - q x_2 x_1)}, \quad \tau \text{ and } q \text{ as before.}$$

4. Truncated quantum affine space:

$$k\langle x_1,\dots,x_n\rangle / (x_i^{n_i},\ x_ix_j - q_{ij}x_jx_i \mid 1 \le i,j \le n)\ ,\quad q_{ij} \text{ as before.}$$

5. Weyl algebra:

$$k[x] \otimes_\tau k[y] \cong k\langle x,y\rangle/(xy-yx-1)\ ,\quad \tau(y\otimes x) = x\otimes y - 1\otimes 1.$$

6. Weyl algebra in $2n$ variables:

$$k\langle x_1,\dots,x_n,y_1,\dots,y_n\rangle/(x_ix_j - x_jx_i,\ y_iy_j - y_jy_i,\ x_iy_j - y_jx_i - \delta_{ij} \mid 1 \le i,j \le n).$$

7. Jordan plane:

$$k[x] \otimes_\tau k[y] \cong k\langle x,y\rangle/(xy-yx-x^2)\ ,\quad \tau(y\otimes x) = x\otimes y - x^2\otimes 1.$$

<u>Examples:</u> (of deformations)

1. Quantum plane or skew polynomial ring:

Let $A = k[x,y]$, define a multiplication $*$ on $A[[t]]$ via:

$$x^i * x^j = x^{i+j}\ ,\quad y^i * y^j = y^{i+j}\ ,\quad x^i * y^j = x^iy^j\ , \text{ and}$$

$$y * x = xy\cdot\left(1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \cdots\right) = xy\cdot\exp(t)$$

as long as the exponential function converges over $k$, or we work formally.

Specializing $t_0 \in k$, so $q = \exp(t_0)$, we obtain:

$$k\langle x,y\rangle/(xy - qyx).$$

2. Quantum affine space: is obtained analogously.

3. Truncated quantum plane:

Let $A = k[x,y]/(x^n, y^m)$, define the same multiplication as before. We obtain:

$$k\langle x,y\rangle/(x^n, y^m, xy - qyx).$$

4. Truncated quantum affine space: is obtained analogously.

5. Weyl algebra:

Let $A = k[x,y]$, define a multiplication $*$ on $A[[t]]$ via:

$$x^i * x^j = x^{i+j}, \quad y^i * y^j = y^{i+j}, \quad x^i * y^j = x^i y^j, \text{ and}$$

$$y * x = xy + t.$$

Specializing to $t=1$, we obtain:

$$k\langle x,y\rangle/(xy - yx - 1)$$

6. Weyl algebra in $2n$ variables: is obtained analogously.

7. Jordan plane:

Let $A = k[x,y]$, define a multiplication $*$ on $A[[t]]$ via:

$$x^i * x^j = x^{i+j}, \quad y^i * y^j = y^{i+j}, \quad x^i * y^j = x^i y^j, \text{ and}$$

$$y * x = xy + x^2 t$$

$$y * x = xy + x^2$$

Specializing to $t=1$, we obtain:

$$k\langle x,y\rangle / (xy - yx - x^2) .$$

## 3. Hochschild cohomology.

An important feature of a deformation is that the multiplication is associative. This imposes conditions on the new multiplication.

Let $(A_t, \mu_t)$ be a deformation of $A$, we want $(a*b)*c = a*(b*c)$ for all $a,b,c \in A$. Applying $a*b = \sum_{i\in\mathbb{N}} \mu_i(a\otimes b)t^i$ and looking at the coefficient of $t$ we obtain:

$$0 = a\,\mu_1(b\otimes c) - \mu_1(ab\otimes c) + \mu_1(a\otimes bc) - \mu_1(a\otimes b)c = d^*(\mu_1)(a\otimes b\otimes c)$$

which looks like a 2-cocycle in some cohomology. Looking at the coefficients of $t^2$ we obtain:

$$\mu_1(\mu_1(a\otimes b)\otimes c) - \mu_1(a\otimes \mu_1(b\otimes c)) = a\,\mu_2(b\otimes c) - \mu_2(ab\otimes c) + \mu_2(a\otimes bc) - \mu_2(a\otimes b)c$$

which looks like a commutator for the left hand side, and again a differential seems to show up on the right hand side. These suspicious are founded!

Consider $A^{\otimes(n+2)}$ as an $A$-bimodule, the bar resolution for $A$ is:

$$\cdots \xrightarrow{d_3} A^{\otimes 4} \xrightarrow{d_2} A^{\otimes 3} \xrightarrow{d_1} A \otimes A \xrightarrow{m_A} A \longrightarrow 0$$

with:

$$d_n(a_0 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1} \quad \text{for all } a_0, \dots, a_{n+1} \in A.$$

Taking the cohomology of this projective resolution gives the Hochschild cohomology of $A$:

$$HH^n(A) := \mathrm{Ext}^n_{A\text{-}A}(A, A) \quad , \qquad HH^*(A) := \bigoplus_{n \in \mathbb{N}} \mathrm{Ext}^n_{A\text{-}A}(A, A)$$

where we are taking coefficients in $A$.

There are two operations natively defined on the cochains of the bar resolution. These provide the Hochschild cohomology with the structure of a graded commutative algebra, and a graded Lie algebra:

Cup product: $\quad \cup : HH^n(A) \times HH^m(A) \longrightarrow HH^{n+m}(A).$

Gerstenhaber bracket: $\quad [-,-] : HH^n(A) \times HH^m(A) \longrightarrow HH^{n+m-1}(A).$

These operations are compatible, providing the Hochschild cohomology with the structure of a Poisson 2-algebra, or Poisson algebra with Poisson bracket of degree $-1$. To fix ideas we can think of these structures as graded Lie algebras coming from an associative algebra, but this intuition is not quite right.

Theorem: [Gerstenhaber] The Hochschild cohomology (together with its multiple structures)
1963-1974

detects the deformations of an algebra.

However, computing it is hard, and in particular computing Gerstenhaber brackets is extremely hard.

## 4. Results.

A multitude of authors have worked on finding the underlying k-vector space of Hochschild cohomologies, and without any claims to exhaustivity, some of which influenced me the most have been Ø. Solberg, M. Van den Bergh, C. Cibils, A. Solotar, C. Negron, S. Witherspoon.

Many others have worked on computing the ring structure arising from the cup product, such as R.-O. Buchweitz, D. Happel.

When looking at the Gerstenhaber bracket, there is a massive difference between being able to compute it by providing formulas, and actually describing the resulting algebra in terms of known objects.

Theorem: [Wambst] Hochschild cohomology for quantum planes.
1993

Theorem: [Buchweitz, Green, Madsen, Solberg] Hochschild cohomology for a truncated
2004

quantum plane : $HH^*\left(\frac{k\langle x,y\rangle}{\dots}\right) \cong k[z]/\dots \times_k \Lambda(u_1,u_2)$

$$\Big( \overline{(x^2, xy + qyx, y^2)} \Big) \qquad \overline{(z^2)}$$

when $q$ is <u>not</u> a root of unity. This provides an example of a finite dimensional algebra with finite Hochschild cohomology that is <u>not</u> of finite global dimension.

<u>Theorem:</u> [Negron, Witherspoon] The Gerstenhaber bracket can be defined directly on the
2015
Koszul resolution (among others).

<u>Theorem:</u> [KMOOW] We provide explicit formulas for the Gerstenhaber bracket of $HH^*(A \otimes_\tau B)$ in terms of $A$ and $B$. We use them to compute the Gerstenhaber algebra structure of the Jordan plane.

<u>Theorem:</u> [OOW] We vastly simplified the proofs of known isomorphisms of Gerstenhaber algebras, such as: $HH^*(A \otimes B) \cong HH^*(A) \otimes HH^*(B)$.

<u>Work in progress:</u> [OV] Finish the classification of twisted tensor planes: $k[x] \otimes_\tau k[y]$, and find an algebra with non-zero $HH^2(A)$ and with <u>no</u> deformations.