The Relative Künneth Theorem

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BRIDGES

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Setup

①

Eilenberg-Moore:

$$\mathcal{C} \underset{F}{\overset{G}{\rightleftarrows}} \mathcal{D}, \qquad A\text{-mod} \;\; \overset{A \otimes_B -}{\longleftarrow} \;\; \longrightarrow \;\; \underset{\mathrm{Hom}_B(A,-)}{\longleftarrow} \;\; B\text{-mod}$$

Hochschild: consider exact sequences with respect to $B \subseteq A$.

Inception of relative homological algebra.

Relative Homological Algebra

②

Let $B \subseteq A$ unital subring.

(A,B)-exact:

$$\cdots \longrightarrow M_i \xrightarrow{d_i} M_{i-1} \longrightarrow \cdots$$

(i) $\mathrm{Ker}(d_i) = \mathrm{im}(d_{i+1})$ ⟵ A-exact.

(ii) $M_i \simeq \mathrm{Ker}(d_i) \oplus Q_i$ in B-mod

Equivalently: $\cdots \longrightarrow M_i \underset{s_i}{\overset{d_i}{\rightleftarrows}} M_{i-1} \longrightarrow \cdots$

(1) Over B-mod we have:

$d_i d_{i+1} = 0$ and $d_{i+1} s_i + s_i d_i = 1_{M_i}$.

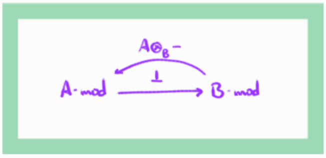
(2) Over B-mod $M_\cdot$ is split exact.

(ii) $M_i = \mathrm{Ker}(d_i) \oplus Q_i$ in B-mod.

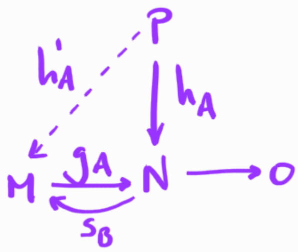
Special Modules

③

(A,B)-free: $A \otimes_B X$, X in B-mod.



(A,B)-projective:



Bottom row is (A,B)-exact.

(*) **(A,B)-flat:** For every (A,B)-exact $0 \to L \to M \to N \to 0$ then:

$$0 \longrightarrow L \otimes_A F \longrightarrow M \otimes_A F \longrightarrow N \otimes_A N \longrightarrow 0$$ is $(\mathbb{Z}, \mathbb{Z})$-exact.

Relative Künneth Theorem

④

Theorem: (Relative Künneth Theorem) Let $(M_\cdot, m_\cdot)$ be a complex of right A-modules in the relative setting. Let $(N_\cdot, n_\cdot)$ be a complex of left A-modules in the relative setting. Then:

$$\bigoplus_{r+s=i} H_r(M_\cdot) \otimes_A H_s(N_\cdot) \rightleftarrows H_i(M_\cdot \otimes_A N_\cdot) \rightleftarrows \bigoplus_{r+s=i-1} \mathrm{Tor}_1^{(A,B)}(H_r(M_\cdot), H_s(N_\cdot))$$

are split short exact sequences of $\mathbb{Z}$-modules.

⑤

Thank you!

EXAMPLES

(Q1)

$$\begin{array}{ccc} \text{free} & \Rightarrow & (A,B)\text{-free} \\ \Downarrow & & \Downarrow \\ \text{projective} & \Rightarrow & (A,B)\text{-projective} \\ \Downarrow & & \Downarrow \\ \text{flat} & \Rightarrow & (A,B)\text{-flat} \end{array}$$

1. $J \subseteq k[x_1, \dots, x_i]$ ideal. Not $(k[x_1, \dots, x_i], k)$-flat.

2. $\mathbb{Z}/(n)$ is $(\mathbb{Z}, \mathbb{Z})$-flat but <u>not</u> $\mathbb{Z}$-flat.

3. A k-algebra, i.d., <u>not</u> field. Q field of fractions is: (A,k)-flat, <u>not</u> (A,k)-projective.

APPLICATION

(Q2)

(*) (A,B)-flat is unusual.

Given $0 \to L \rightleftarrows M \rightleftarrows N \to 0$ (A,B)-exact:

<u>F (A,B)-flat:</u>

$$0 \to L \otimes_A F \rightleftarrows M \otimes_A F \rightleftarrows N \otimes_A F \to 0 \text{ is } (\mathbb{Z},\mathbb{Z})\text{-exact.}$$

<u>F "relatively flat":</u> Weibel

$$0 \to L \otimes_A F \to M \otimes_A F \to N \otimes_A F \to 0 \text{ is exact.}$$

<u>Proposition:</u> F is (A,B)-flat $\Leftrightarrow$ F is relatively flat.

(Q3)

<u>Theorem:</u> The following are equivalent:

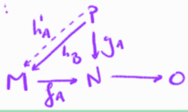
(1) P is (A,B)-projective.

(2) Every (A,B)-exact sequence $0 \to M \to N \to P \to 0$ splits as A-modules

splits as A-modules.

(3) P is a direct summand of an (A,B)-free module.

(4) We can complete the diagram:



Remark:

(A,B)-flat modules preserve (A,B)-exact sequences:

$(M_\cdot, d_\cdot)$ left (A,B)-exact then

$(M_\cdot \otimes_A F, d_\cdot \otimes 1_F)$ is $(\mathbb{Z},\mathbb{Z})$-exact.

Theorem: The following are equivalent:

(1) F is (A,B)-flat.

(2) $\mathrm{Tor}_i^{(A,B)}(M,F) = 0$ for all A-modules M and $i \in \mathbb{N}$.

(3) $\mathrm{Tor}_1^{(A,B)}(M,F) = 0$ for all A-modules M.