# Hochschild Cohomology of Twisted Tensor Product Algebras

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## Motivation

### Theorem (Le-Zhou) [3]

Consider $A$ and $B$ two algebras over $k$ a field. When considered with the correct structures, we have:

$$HH^*(A \otimes B) \cong HH^*(A) \otimes HH^*(B).$$

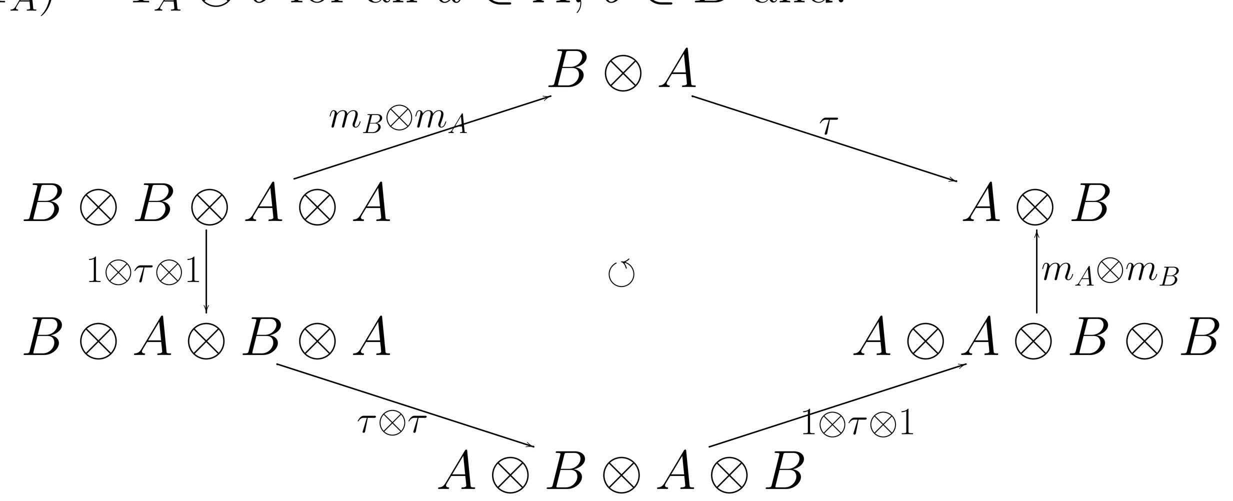
This involves understanding the Gerstenhaber algebra structure on $HH^*(A\otimes B)$ and providing a suitable structure to $HH^*(A)\otimes HH^*(B)$.

### Goal

We want to understand $HH^*(A \otimes_\tau B)$ in terms of $HH^*(A)$ and $HH^*(B)$. More specifically, given a resolution of $A$ as $A^e$ module, and a resolution of $B$ as $B^e$ module, we compute a resolution of $A \otimes_\tau B$ as $(A \otimes_\tau B)^e$ module.

## Setup [1] [2] [4]

Let $A$, $B$ two algebras over $k$. We say that a bijective $k$ linear map $\tau : B \otimes A \longrightarrow A \otimes B$ is a twisting map if $\tau(1_B \otimes a) = a \otimes 1_B$ and $\tau(b \otimes 1_A) = 1_A \otimes b$ for all $a \in A$, $b \in B$ and:




Under this condition:

- the twisted tensor product algebra $A \otimes_\tau B$ is the vector space $A \otimes B$ with multiplication induced by $\tau$,
- an $A$ bimodule $M$ is compatible with $\tau$ if there exists a bijective $k$ linear map $\tau_{B,M} : B \otimes M \longrightarrow M \otimes B$ that is well behaved with respect to the algebra structure of $B$ and the module structure of $M$.

### Proposition

Given $M$ and $N$ bimodules over $A$ and $B$ compatible with $\tau$, then $M \otimes N$ has a natural bimodule structure over $A \otimes_\tau B$.

## Compatibility of resolutions

Given $M$ an $A$ bimodule that is compatible with $\tau$, we say that a projective $A^e$ resolution $P_\bullet(M)$ is compatible with $\tau$ if each $P_i(M)$ is compatible with $\tau$ via a map $\tau_{B,i} : B \otimes P_i(M) \longrightarrow P_i(M) \otimes B$ such that $\tau_{B,\bullet}$ is a chain map lifting $\tau_{B,M}$.

### Proposition [4]

Let $\tau$ be a twisting map for $A$ and $B$. Then $\mathbb{B}(A)$ and $\mathbb{B}(B)$, the bar resolutions of $A$ and $B$ respectively, are compatible with $\tau$.

## Technique(s)

We need to define the maps guaranteeing compatibility. For each $n \in \mathbb{N}$ define the maps $\tau_{B,n} : B \otimes \mathbb{B}_n(A) \longrightarrow \mathbb{B}_n(A) \otimes B$ recursively:

$$\tau_{B,0} := 1 \otimes \tau \circ \tau \otimes 1, \tau_{B,n} := 1 \otimes \tau \circ \tau_{B,n-1} \otimes 1.$$

Through this iteration of $\tau$ we have that $A$ and $B$ satisfy the prerequisites of compatibility. This is proven via commutative diagrams.

### Proposition

❶ Commutativity with the product in $B$:

$$\tau_{B,n} \circ m_B \otimes 1 = 1 \otimes m_B \circ \tau_{B,n} \otimes 1 \circ 1 \otimes \tau_{B,n}.$$

❷ Commutativity with the bimodule structure:

$$\tau_{B,n} \circ 1 \otimes \rho_{A,n} = \rho_{A,n} \otimes 1 \circ 1 \otimes 1 \otimes \tau \circ 1 \otimes \tau_{B,n} \otimes 1 \circ \tau \otimes 1 \otimes 1.$$

❸ Lifting to a chain map:

$$\tau_{B,n+1} \circ 1 \otimes d_n = d_n \otimes 1 \circ \tau_{B,n+2}.$$

Analogous results for $A$ and $\mathbb{B}(B)$ follow. In fact, when it exists, we can descend these equalities to the Koszul resolution.

### Proposition

Let $\tau$ be a twisting map for $A$ and $B$ Koszul algebras with $\mathbb{K}(A)$ and $\mathbb{K}(B)$ their respective Koszul resolutions. Then $\mathbb{B}(A)$ or $\mathbb{K}(A)$ and $\mathbb{B}(A)$ or $\mathbb{K}(B)$ are compatible with $\tau$.

Since the module structures are given via $\tau$, we need to be careful about what diagrams are indeed commutative: we are using the non-commutative analogue of the usual tensor product, and so permuting its components as we would usually do is not allowed.

## Key idea(s)

Seeing the Koszul resolution as lying entirely inside the bar resolution we can use our reasoning over diagrams to generalize a result (for graded algebras) in [2] that gives an explicit formula for the Gerstenhaber bracket of $HH^*(R\otimes^t S)$ in terms of elements in $HH^*(R)$ and $HH^*(S)$.

### Proposition

Resolving $A$ and $B$ via the bar or the Koszul resolutions, we can explicitly compute the Gerstenhaber bracket of $HH^*(A \otimes_\tau B)$ in terms of elements in $HH^*(A)$ and $HH^*(B)$.

## Applications [2] [5]

- **Truncated algebras**: quantum complete intersections arising from homomorphisms $t : A \otimes_{\mathbb{Z}} B \longrightarrow k^\times$, treatable for some $q \in k^\times$:

$$k\langle x, y\rangle/(x^2, y^2, xy + qyx).$$

- **Jordan plane**: arising from the map $\tau : k[y] \otimes k[x] \to k[x] \otimes k[y]$ given by $\tau(y \otimes x) = x \otimes y + x^2 \otimes 1$:

$$k\langle x, y\rangle/(yx - xy - x^2).$$

## References

[1] A. Čap, H. Schichl and J. Vanžura, *On twisted tensor products of algebras*, Comm. Algebra 23 (1995), no. 12, 4701–4735.

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