Homotopy liftings and Hochschild cohomology of some twisted tensor products

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Abstract

The (ring structure of the) Hochschild cohomology of the tensor product of two algebras was understood better thanks to Le and Zhou, who were able to express it in terms of the Hochschild cohomology of the two algebras. Using work by Grimley, Nguyen, and Witherspoon, as well as homotopy lifting techniques for Gerstenhaber brackets introduced by Volkov, we generalize Le and Zhou's result to some twisted tensor products. These have important applications in some quantum complete intersections also studied by Lopes and Solotar. This is joint work with Tolulope Oke and Sarah Witherspoon.

Contents

1 Motivation

Before jumping into the definitions and the abstract concepts, let me tell you about the result motivating some of this work: In 2014 Le and Zhou proved that $\mathrm{HH}^*(A \otimes B) \cong \mathrm{HH}^*(A) \otimes \mathrm{HH}^*(B)$ under some finiteness conditions. It concerns *Hochschild cohomology* and the usual tensor product of k-algebras, which is commutative. There are generalizations of this tensor product to the non-commutative case, like the *twisted tensor product*, and it would be nice to have a similar behavior over it, something like "$\mathrm{HH}^*(A \otimes_\tau B) \cong \mathrm{HH}^*(A) \otimes_\tau \mathrm{HH}^*(B)$". Unfortunately, that symbol-by-symbol translation cannot possibly be correct: the left hand side is a graded commutative algebra, while the right hand side is (in general) a non-commutative algebra.

What is the correct translation of Le and Zhou's result to the non-commutative case? What techniques does it involve? Here we will attempt to address partially some of these questions.

2 Hochschild cohomology

Definition 1. *Let A be a k-algebra (our algebras are unital and associative, I'm not a monster. We define the Hochschild cohomology as):* $\mathrm{HH}^n(A) = \mathrm{Ext}^n_{A^e}(A, A)$ *where* $A^e = A \otimes A^{op}$ *(is called* *the enveloping algebra of A). It comes with two operations (defined on cochains):*

$$\smile : \mathrm{HH}^m(A) \times \mathrm{HH}^n(A) \longrightarrow \mathrm{HH}^{m+n}(A),$$
$$[-,-] : \mathrm{HH}^m(A) \times \mathrm{HH}^n(A) \longrightarrow \mathrm{HH}^{m+n-1}(A).$$

We call $\smile$ the *cup product* and $[-,-]$ the *Gerstenhaber bracket*. The cup product gives $(\mathrm{HH}^*(A), \smile)$ the structure of a graded commutative algebra. The Gerstenhaber bracket gives $(\mathrm{HH}^*(A), [-,-])$ the structure of a graded Lie algebra. Together with some compatibility conditions, they give $(\mathrm{HH}^*(A), \smile, [-,-])$ the structure of a Gerstenhaber algebra, also known as Poisson 2-algebra, or a Poisson algebra with Poisson bracket of degree -1.

To fix ideas, this last structure can be thought of as a graded Lie algebra coming from an associative algebra. There are some key differences, since to be precise the degree of $\mathrm{HH}^*(A)$ with respect to $[-,-]$ (also called the Lie degree) is one less than the degree of $\mathrm{HH}^*(A)$ with respect to $\smile$ (also called the homological degree).

Theorem 2 (Le-Zhou 2014)**.** *Let A and B be k-algebras, at least one of them finite dimensional. Then (as Gerstenhaber algebras):*

$$\mathrm{HH}^*(A \otimes B) \cong \mathrm{HH}^*(A) \otimes \mathrm{HH}^*(B).$$

Proof. Working over the bar resolution and using the cumbersome Alexander-Whitney and Eilenberg-Zilber maps. □

You may complain, and rightfully so, that I have not told you how one can get a cup product or a Gerstenhaber bracket on a tensor product of Gerstenhaber algebras. For now, suffice to say that they exist and they satisfy what they should satisfy. The explicit expressions are:

$$(\alpha \otimes \beta) \smile (\alpha' \otimes \beta') = (-1)^{m'n}(\alpha \smile \alpha') \otimes (\beta \smile \beta'),$$
$$[\alpha \otimes \beta, \alpha' \otimes \beta'] = (-1)^{(m'-1)n}[\alpha, \alpha'] \otimes (\beta \smile \beta') + (-1)^{m'(n-1)}(\alpha \smile \alpha') \otimes [\beta, \beta'].$$

3 Twisted tensor product of algebras

Definition 3. *Let A and B be k-algebras, a twisting map $\tau : B \otimes A \to A \otimes B$ is a bijective k-linear map (with the conditions $\tau(1_B \otimes a) = a \otimes 1_B$, $\tau(b \otimes 1_A) = 1_A \otimes b$ for all $a \in A$, $b \in B$, and:*

$$\tau \circ (m_B \otimes m_A) = (m_A \otimes m_B) \circ (1 \otimes \tau \otimes 1) \circ (\tau \otimes \tau) \circ (1 \otimes \tau \otimes 1)$$

meaning that twisting and multiplication "commute". Equivalently

$$\begin{array}{ccccc} B \otimes B \otimes A \otimes A & \xrightarrow{1\otimes\tau\otimes 1} & B \otimes A \otimes B \otimes A & \xrightarrow{\tau\otimes\tau} & A \otimes B \otimes A \otimes B \\ \downarrow{\scriptstyle m_B\otimes m_A} & & & & \downarrow{\scriptstyle 1\otimes\tau\otimes 1} \\ B \otimes A & \xrightarrow{\quad\tau\quad} & A \otimes B & \xleftarrow{m_A\otimes m_B} & A \otimes A \otimes B \otimes B \end{array}$$

is a commutative diagram). The twisted tensor algebra $A \otimes_\tau B$ is $A \otimes B$ (as a vector space) with (as it turns out associative) multiplication:

$$m_{A\otimes_\tau B} : A \otimes B \otimes A \otimes B \xrightarrow{1\otimes\tau\otimes 1} A \otimes A \otimes B \otimes B \xrightarrow{m_A\otimes m_B} A \otimes B.$$

Of course, the idea of working with $\mathrm{HH}^*(A \otimes_\tau B)$ is not to treat $A \otimes_\tau B$ as a given algebra, but to use some information about A and B that is already known. For example, if we have additional information about cocycles, or we know the Gerstenhaber bracket in the respective Hochschild cohomologies of A and B, then a combination of Lemma 8 and Theorem 7 allows computing the Gerstenhaber bracket in the Hochschild cohomology of a twisted tensor product by a bicharacter, a notoriously difficult task.

Example 4. *Let A, B be k-algebras graded by the commutative groups F, G respectively, let $t : F \otimes_{\mathbb{Z}} G \to k^\times$ be a bicharacter. Then $\tau(b \otimes a) = t(|a|, |b|)a \otimes b$ is a twisting map, we denote $A \otimes^t B = A \otimes_\tau B$. It can be checked that $\mathrm{HH}^*(-)$ is bigraded: $\mathrm{HH}^{*,*}(-)$. We denote:*

$$F' = \bigcap_{g \in G} \ker(t(-, g)), \quad G' = \bigcap_{f \in F} \ker(t(f, -)).$$

Theorem 5 (Grimley-Nguyen-Witherspoon 2017, OOW)**.** *As Gerstenhaber algebras (in the twisted tensor product setup, and assuming the necessary finiteness conditions, we have):*

$$\mathrm{HH}^{*,F' \oplus G'}(A \otimes^t B) \cong \mathrm{HH}^{*,F'}(A) \otimes \mathrm{HH}^{*,G'}(B).$$

Proof. (The original proof used extended versions of the Alexander-Whitney and Eilenberg-Zilber maps. We completely avoided them by using) Volkov's homotopy lifting (techniques, as well as a chain isomorphism, and a bit of work with the Koszul sign convention). ☐

What are Volkov's homotopy liftings? They are nice chain maps between shifted resolutions.

Definition 6 (Volkov 2016)**.** *(Given A a k-algebra,) let $\mu_P : P \to A$ be a resolution of A-bimodules, $\Delta_P : P \to P \otimes_A P$ a diagonal map, and $\alpha \in \hom_{A^e}(P_m, A)$ a cocycle. A homotopy lifting (of α with respect to Δ_P) is (an A-bimodule chain homomorphism) $\psi_\alpha : P \to P[1-m]$ satisfying (some very) technical conditions (depending only on the augmentation map μ_P, the diagonal map Δ_P, and the cocycle α:*

$$d(\psi_\alpha) = (\alpha \otimes 1_P - 1_P \otimes \alpha)\Delta_P, \quad \text{and} \quad \mu_P \psi_\alpha \text{ is cohomologous to } (-1)^{m-1}\alpha\psi$$

for some A-bimodule chain map $\psi : P \to P[1]$ for which $d(\psi) = (\mu_P \otimes 1_P - 1_P \otimes \mu_P)\Delta_P$).

The definition of homotopy lifting does not depend on a specific resolution, since such diagonal maps always exist. Moreover, Volkov proved that for any resolution, for any diagonal, and for any cocycle, homotopy liftings always exist! Moreover, they induce the Gerstenhaber bracket in cohomology! This is absolutely fantastic.

Theorem 7 (Volkov 2016)**.** *The bracket (given at the chain level by):*

$$[\alpha, \beta] = \alpha\psi_\beta - (-1)^{(m-1)(n-1)}\beta\psi_\alpha$$

induces the Gerstenhaber bracket (on Hochschild cohomology).

This method is inspired in results and work by Negron and Witherspoon, who in 2016 published what now is a special case of these homotopy liftings, where they focused on Koszul-like resolutions. We found explicit homotopy liftings for the twist by a bicharacter.

Lemma 8 (OOW)**.** *In the twisted tensor product setup, let $P \to A$ and $Q \to B$ be resolutions of algebras (with the necessary finiteness conditions):*

$$\psi_{\alpha \otimes^t \beta} = \psi_\alpha \otimes^t (1_Q \otimes_B \beta)\Delta_Q + (-1)^m(\alpha \otimes_A 1_P)\Delta_P \otimes^t \psi_\beta$$

is a homotopy lifting of $\alpha \otimes^t \beta$ (in terms of homotopy liftings of α and β).

4 Applications and future work

The result by [LZ] can be obtained by mimicking our proof of Theorem 5 with a homotopy lifting closely resembling the one of Lemma 8. This suggests that although finding them may seem like a daunting task, in practice this may be feasible. Many other results are expected to allow a similar proof by these techniques, including some that are not yet known to be true.

In [GNW] they computed the Gerstenhaber algebra structure of the quantum complete intersections $\Lambda_q = k\langle x, y\rangle/(x^2, y^2, xy + qyx)$ for $q \in k^\times$ using the techniques they developed to prove Theorem 5. They found that in many cases $\mathrm{HH}^1(\Lambda_q)$ is a finite dimensional abelian Lie algebra over which $\mathrm{HH}^*(\Lambda_q)$ is a module (the generators being common eigenvectors). One of the exceptions was $q = 1$ and $\mathrm{char}(k) \neq 2$, where $\mathrm{HH}^1(\Lambda_1)$ is isomorphic to the Lie algebra $\mathfrak{gl}_2(k)$. It still acts on $\mathrm{HH}^*(\Lambda_1)$, but in a more complicated way.

The Jordan plane: $k\langle x, y\rangle/(yx-xy-x^2)$ can be seen as $k[x]\otimes_\tau k[y]$ for $\tau(y\otimes x) = x\otimes y+x^2\otimes 1$. The complete Gerstenhaber algebra structure of the Jordan plane was first computed by Lopes and Solotar, using spectral sequences and a lot of machinery. Our computations in [KMOOW] used more elementary and completely different methods, enabled by Volkov's homotopy lifting techniques.

The use of Volkov's homotopy liftings seems to be the way of tackling this type of problems, and they should enable elementary computations of examples like Λ_q.

Thank you for your time!

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