Homotopy liftings and Hochschild cohomology of some twisted tensor products.

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Contents:

1. Motivation.

usual tensor product of algebras: $A \otimes B \cong B \otimes A$.

Le and Zhou proved: $\boxed{HH^*(A \otimes B) \cong HH^*(A) \otimes HH^*(B)}$

usual Hochschild cohomology

There is a non-commutative tensor product of algebras: $\otimes_\tau$.

We would like to have: $\boxed{HH^*(A \otimes_\tau B) \cong HH^*(A) \otimes_\tau HH^*(B)}$.

⚠ unfortunately, not true.

What is the correct interpretation?

2. Hochschild cohomology.

Definition: Let A be a k-algebra, $A^e := A \otimes A^{op}$.

$$\boxed{\begin{aligned} HH^n(A) &:= \mathrm{Ext}^n_{A^e}(A,A), \\ HH^*(A) &:= \bigoplus_{n\in\mathbb{N}} \mathrm{Ext}^n_{A^e}(A,A). \end{aligned}}$$

Cup product: $\cup : HH^m(A) \times HH^n(A) \longrightarrow HH^{m+n}(A)$.

Gerstenhaber bracket: $[-,-] : HH^m(A) \times HH^n(A) \longrightarrow HH^{m+n-1}(A)$.

$HH^*(A)$ →
- $\cup$: graded commutative algebra.
- $[-,-]$: graded Lie algebra.
- $\cup, [-,-]$: Gerstenhaber algebra.

Theorem: [Le-Zhou] Let A, B be k-algebras, at least one of them finite dimensional. Then:

$$HH^*(A\otimes B) \cong HH^*(A) \otimes HH^*(B) \text{ as Gerstenhaber algebras.}$$

3. Twisted tensor products.

Definition: A twisting map $\tau : B \otimes A \longrightarrow A \otimes B$ is a bijective k-linear map satisfying:

$\tau(1_B \otimes a) = a \otimes 1_B$, $\tau(b \otimes 1_A) = 1_A \otimes b$ for all $a \in A$, $b \in B$,

$$\begin{tikzcd} B\otimes B\otimes A\otimes A \arrow[r, "1\otimes\tau\otimes 1"] \arrow[d, "m_B\otimes m_A"'] & B\otimes A\otimes B\otimes A \arrow[r, "\tau\otimes\tau"] \arrow[d, phantom, "\circlearrowright"] & A\otimes B\otimes A\otimes B \arrow[d, "1\otimes\tau\otimes 1"] \\ B\otimes A \arrow[r, "\tau"] & A\otimes B & A\otimes A\otimes B\otimes B \arrow[l, "m_A\otimes m_B"'] \end{tikzcd}$$

multiply

twist

twist

multiply

The twisted tensor algebra $A\otimes_\tau B$ is $A\otimes B$ with multiplication:

$$m_{A\otimes_\tau B} : A\otimes B\otimes A\otimes B \xrightarrow{1\otimes\tau\otimes 1} A\otimes A\otimes B\otimes B \xrightarrow{m_A\otimes m_B} A\otimes B.$$

Example: twisting by a bicharacter.

Let A, B be k-algebras graded by F, G commutative groups.

Let $t: F \otimes_{\mathbb{Z}} G \longrightarrow k^\times$ be a bicharacter. Then $\tau: B \otimes A \longrightarrow A \otimes B$
$b \otimes a \mapsto t(|a|, |b|)\, a \otimes b$

induces $A \otimes_\tau B =: \boxed{A \otimes^t B}$.

This is $A \otimes B$ as k-vector space and: $(a \otimes b) \cdot^t (a' \otimes b') = t(|a'|, |b|)\, aa' \otimes bb'$.

Here $HH^*(-)$ is bigraded: $HH^{*,*}(-)$.

Theorem: [Grimley-Nguyen-Witherspoon, OOW] Under some finiteness conditions, we have:

$$\boxed{HH^{*, F' \otimes G'}(A \otimes^t B) \cong HH^{*,F'}(A) \otimes HH^{*,G'}(B)}$$ as Gerstenhaber algebras.

Where:

$$F' := \bigcap_{g \in G} \ker(t(-,g)), \quad G' := \bigcap_{f \in F} \ker(t(f,-)).$$

Proof: Original: cumbersome.

New: avoids cumbersomeness by using Volkov's homotopy liftings. $\square$

Theorem: [Volkov] The bracket given at the chain level by:

$$\boxed{[\alpha, \beta] := \alpha \psi_\beta - (-1)^{(|\alpha|-1)(|\beta|-1)} \beta \psi_\alpha}$$

induces the Gerstenhaber bracket on $HH^*(-)$.

ψ_α, ψ_β are homotopy liftings of α, β.

They always exist.

This result was inspired by [Negron-Witherspoon].

Lemma: [OOW] Let $P \to A$, $Q \to B$ be resolutions, then:

Lemma: [OOW] Let $P \to A$, $Q \to B$ be resolutions, then:

$$\boxed{\psi_{\alpha \otimes^t \beta} := \psi_\alpha \otimes^t (1_Q \otimes_B \beta)\Delta_Q + (-1)^{|\alpha|}(\alpha \otimes_A 1_P)\Delta_P \otimes^t \psi_\beta}$$

is a homotopy lifting of $\alpha \otimes^t \beta$.

4. Applications and future work.

Le and Zhou's result follows with a similar homotopy lifting. What else?

Quantum complete intersections [Grimley-Nguyen-Witherspoon]:

$$k\langle x,y\rangle / (x^2, y^2, xy + qyx), \quad q \in k^\times.$$

Jordan plane [Lopes-Solotar, Karadag-McPhate-O-Oke-Witherspoon]:

$$k\langle x,y\rangle / (yx - xy - x^2), \quad k[x] \otimes_\tau k[y] \text{ with } \tau(y \otimes x) = x \otimes y + x^2 \otimes 1.$$

More computations!

Thank you!