Twisted tensor product algebras and compatibility of the bar resolution

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March 2, 2019

Outline

Why do we care?

1. Sometimes we can understand the (co)homology theory of a tensor product in terms of the (co)homology of the original factors.
2. This understanding relies on the tensor product of projective resolutions for the factor algebras being a projective resolution for the tensor product of the algebras.
3. Čap, Schichl, and Vanžura introduced twisted tensor products in 1995 as an analogue for non commutative algebras.
4. In concrete settings, a construction similar to the commutative case have been achieved, yielding similar results.
5. Shepler and Witherspoon unify many of these constructions in 2018.

Computational use in concrete settings

1. Negron and Witherspoon in 2016 develop techniques to construct Gerstenhaber brackets on Hochschild cohomology.
2. Grimley, Nguyen and Witherspoon augmented these techniques in 2017, constructing and computing the Gerstenhaber bracket in some twisted tensor products.
3. Can these conditions be relaxed to compute the Gerstenhaber bracket of a twisted tensor product? If so, how much?

Algebras over a ring (I)

Definition

Let k be an associative commutative ring. We say that A is a k *algebra* if it is a k module and a ring, where the product $\mu : A \times A \longrightarrow A$ is bilinear.

Examples:

- Commutative: $k[x]$, $k[x_1, \ldots, x_n]$, $k[x]/(x^n)$ for $n \in \mathbb{N}$.
- Noncommutative: $k\langle x, y\rangle/(yx - xy - x^2)$.

Definition

Let A be a k algebra. We define A^{op} the *opposite algebra of* A as the vector space A with multiplication $\mu_{op} : A \times A \longrightarrow A$ given by:

$$\mu_{op}(a, b) = \mu(b, a) \text{ for all } a, b \in A.$$

Algebras over a ring (and II)

Definition

Let A be a k algebra. We define A^e the *enveloping algebra of A* as the vector space $A \otimes A^{op}$ with multiplication $\mu^e : A^e \times A^e \longrightarrow A^e$ given by:

$$\mu^e((a_1 \otimes b_1), (a_2 \otimes b_2)) = \mu(a_1, a_2) \otimes \mu_{op}(b_1, b_2) = a_1 a_2 \otimes b_2 b_1$$

for all $a_1, a_2, b_1, b_2 \in A$.

Examples:

- $k[x]^e = k[x] \otimes k[y] \cong k[x, y]$.
- $k[x]/(x^n)^e = k[x]/(x^n) \otimes k[y]/(y^n) \cong k[x, y]/(x^n, y^n)$ for $n \in \mathbb{N}$.

For technical reasons, from now on we take k to be a field.

Twisted tensor product algebra

Definition

Let A, B two algebras over k. We say that a bijective k linear map $\tau : B \otimes A \longrightarrow A \otimes B$ is a *twisting map* if $\tau(1_B \otimes a) = a \otimes 1_B$ and $\tau(b \otimes 1_A) = 1_A \otimes b$ for all $a \in A$, $b \in B$ and:

$$\begin{tikzcd}
B \otimes B \otimes A \otimes A \arrow[r, "m_B \otimes m_A"] \arrow[d, "1\otimes\tau\otimes 1"'] & B \otimes A \arrow[rr, "\tau"] \arrow[d, phantom, "\circlearrowleft"] & & A \otimes B \\
B \otimes A \otimes B \otimes A \arrow[r, "\tau\otimes\tau"] & A \otimes B \otimes A \otimes B \arrow[rr, "1\otimes\tau\otimes 1"] & & A \otimes A \otimes B \otimes B \arrow[u, "m_A \otimes m_B"']
\end{tikzcd}$$

Definition

Under this condition, the *twisted tensor product algebra* $A \otimes_\tau B$ is the vector space $A \otimes B$ with multiplication:

$$m_\tau : (A \otimes B) \otimes (A \otimes B) \xrightarrow{1\otimes\tau\otimes 1} A \otimes A \otimes B \otimes B \xrightarrow{m_A \otimes m_B} A \otimes B$$

Bimodule compatible with the twisting (I)

Definition

We say that an A bimodule M, whose bimodule structure is given by $\rho_A : A \otimes M \otimes A \longrightarrow M$, is *compatible with* τ if there exist a bijective k linear map $\tau_{B,M} : B \otimes M \longrightarrow M \otimes B$ such that:

1. $\tau_{B,M}$ is well behaved with respect to the algebra structure of B,
2. the module structure of M is well behaved (via $\tau_{B,M}$) with respect to the algebra structure of B and the twisting map τ.

We analogously define how a B bimodule N is compatible with τ via $\tau_{N,A}$.

Bimodule compatible with the twisting (and II)

$$
\begin{tikzcd}
B \otimes B \otimes M \arrow[r, "1\otimes\tau_{B,M}"] \arrow[d, "m_B\otimes 1"'] & B \otimes M \otimes B \arrow[r, "\tau_{B,M}\otimes 1"] & M \otimes B \otimes B \arrow[d, "1\otimes m_B"] \\
B \otimes M \arrow[rr, "\tau_{B,M}"] & & M \otimes B \\
B \otimes A \otimes M \otimes A \arrow[u, "1\otimes\rho_A"] \arrow[d, "\tau\otimes 1\otimes 1"'] & & A \otimes M \otimes A \otimes B \arrow[u, "\rho_A\otimes 1"'] \\
A \otimes B \otimes M \otimes A \arrow[rr, "1\otimes\tau_{B,M}\otimes 1"] & & A \otimes M \otimes B \otimes A \arrow[u, "1\otimes 1\otimes\tau"']
\end{tikzcd}
$$

Twisted bimodule structure of the tensor product

If M and N are A and B bimodules via ρ_A and ρ_B compatible with τ via $\tau_{B,M}$ and $\tau_{N,A}$ respectively, then:

$$\begin{tikzcd}
(A \otimes_\tau B) \otimes (M \otimes N) \otimes (A \otimes_\tau B) \arrow[rr, "\rho_{A\otimes_\tau B}"] \arrow[d, "1\otimes\tau_{B,M}\otimes\tau_{N,A}\otimes 1"'] \arrow[drr, phantom, "\circlearrowleft"] & & M \otimes N \\
A \otimes M \otimes B \otimes A \otimes N \otimes B \arrow[r, "1\otimes 1\otimes\tau\otimes 1\otimes 1"] & A \otimes M \otimes A \otimes B \otimes N \otimes B \arrow[ur, "\rho_A\otimes\rho_B"'] &
\end{tikzcd}$$

defines a natural structure of $A \otimes_\tau B$ bimodule over $M \otimes N$ via $\rho_{A\otimes_\tau B}$.

Compatibility of resolutions (I)

Let $P_\bullet(M)$ be an A^e projective resolution of M and $P_\bullet(N)$ a B^e projective resolution of N. Consider the complexes $P_\bullet(N) \otimes A$, $A \otimes P_\bullet(N)$, $P_\bullet(M) \otimes B$, $B \otimes P_\bullet(M)$.

$$\cdots \longrightarrow P_2(M) \longrightarrow P_1(M) \longrightarrow P_0(M) \longrightarrow M \longrightarrow 0,$$
$$\cdots \longrightarrow P_2(N) \longrightarrow P_1(N) \longrightarrow P_0(N) \longrightarrow N \longrightarrow 0.$$

As exact sequences of vector spaces any k linear maps $\tau_{N,A} : N \otimes A \longrightarrow A \otimes N$ and $\tau_{B,M} : B \otimes M \longrightarrow M \otimes B$ can be lifted to k linear chain maps:

$$\tau_{P_\bullet(N),A} : P_\bullet(N) \otimes A \longrightarrow A \otimes P_\bullet(N), \quad \tau_{B,P_\bullet(M)} : B \otimes P_\bullet(M) \longrightarrow P_\bullet(M) \otimes B,$$

denoted by $\tau_{i,A} := \tau_{P_i(N),A}$ and $\tau_{B,i} := \tau_{B,P_i(M)}$.

Compatibility of resolutions (and II)

Definition

Given M an A bimodule that is compatible with τ, we say that a projective A^e resolution $P_\bullet(M)$ is *compatible with* τ if each $P_i(M)$ is compatible with τ via a map $\tau_{B,i} : B \otimes P_i(M) \longrightarrow P_i(M) \otimes B$ such that $\tau_{B,\bullet}$ is a chain map lifting $\tau_{B,M}$.

Given N a B bimodule compatible with τ, we can analogously define how a projective B^e resolution $P_\bullet(N)$ is compatible with τ via $\tau_{\bullet,A}$.

Special modules and bimodules over an algebra

Remark

A k algebra A is a left A^e module under:

$$(a \otimes b) \cdot c = acb \text{ for all } a, b, c \in A.$$

In particular $HH^\bullet(A) := HH^\bullet(A, A)$ is well defined.

Remark

The tensor product $A^{\otimes n} = A \otimes \overset{(n)}{\cdots} \otimes A$ is is a left A^e module under:

$$(a \otimes b) \cdot (c_1 \otimes c_2 \cdots \otimes c_{n-1} \otimes c_n) = ac_1 \otimes c_2 \cdots \otimes c_{n-1} \otimes c_n b$$

for all $a, b, c_1, \ldots, c_n \in A$.

The bar resolution

Consider the sequence of left A^e modules:

$$\cdots \xrightarrow{d_3} A^{\otimes 4} \xrightarrow{d_2} A^{\otimes 3} \xrightarrow{d_1} A \otimes A \xrightarrow{\mu} A \longrightarrow 0$$

with:

$$d_n(a_0 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}$$

for all $a_0, \ldots, a_{n+1} \in A$. This is a complex by direct computation. It has a contracting homotopy $s_n : A^{\otimes(n+2)} \longrightarrow A^{\otimes(n+3)}$:

$$s_n(a_0 \otimes \cdots \otimes a_{n+1}) = 1 \otimes a_0 \otimes \cdots \otimes a_{n+1}$$

so the complex is exact. Moreover since $A^{\otimes n} \cong \bigoplus_{i \in I} k\alpha_i$ as k modules:

$$A^{\otimes(n+2)} \cong A^e \otimes A^{\otimes n} \cong \bigoplus_{i \in I} A^e(1 \otimes 1 \otimes \alpha_i)$$

so $A^{\otimes(n+2)}$ are free A^e modules, and the complex is a free resolution.

The bar resolution is compatible with the twisting

Proposition

Let τ be a twisting map for the algebras A and B. Then $\mathbb{B}(A)$ and $\mathbb{B}(B)$, the bar resolutions of A and B respectively, are compatible with τ.

We need to say via which maps.

Definition

For each $n \in \mathbb{N}$ define the maps $\tau_{B,n} : B \otimes \mathbb{B}_n(A) \longrightarrow \mathbb{B}_n(A) \otimes B$ recursively: $\tau_{B,0} := 1 \otimes \tau \circ \tau \otimes 1$, $\tau_{B,n} := 1 \otimes \tau \circ \tau_{B,n-1} \otimes 1$.

We define analogously $\tau_{n,A}$. Notice that $\tau_{B,n}$ also satisfies $\tau_{B,0} := 1 \otimes \tau \circ \tau \otimes 1$, $\tau_{B,n} := 1 \otimes \tau_{B,n-1} \circ \tau \otimes 1$.

Proof.

Both A and B satisfy the prerequisites of compatibility necessary to ask whether $\mathbb{B}(A)$ and $\mathbb{B}(B)$ may be compatible with τ.
To see that $\mathbb{B}(A)$ is compatible with τ we need that for all $n \in \mathbb{N}$:

1. Commutativity with the product in B:

$$\tau_{B,n} \circ m_B \otimes 1 = 1 \otimes m_B \circ \tau_{B,n} \otimes 1 \circ 1 \otimes \tau_{B,n}.$$

2. Commutativity with the bimodule structure:

$$\tau_{B,n} \circ 1 \otimes \rho_{A,n} = \rho_{A,n} \otimes 1 \circ 1 \otimes 1 \otimes \tau \circ 1 \otimes \tau_{B,n} \otimes 1 \circ \tau \otimes 1 \otimes 1.$$

3. Lifting to a chain map:

$$\tau_{B,n+1} \circ 1 \otimes d_n = d_n \otimes 1 \circ \tau_{B,n+2}.$$

The second part of the statement follows analogously. □

Compatibility of A and B as bimodules (I)

Lemma

Let τ be a twisting map for the algebras A and B. Then A and B, seen as an A^e module and a B^e module respectively, are compatible with τ.

Proof.

To check the commutativity of the diagram:

$$\begin{tikzcd}
B \otimes B \otimes A \arrow[r, "1\otimes\tau"] \arrow[d, "m_B\otimes 1"'] & B \otimes A \otimes B \arrow[r, "\tau\otimes 1"] & A \otimes B \otimes B \arrow[d, "1\otimes m_B"] \\
B \otimes A \arrow[rr, "\tau"] & & A \otimes B
\end{tikzcd}$$

we just have to set 1_A the identity element of A in the definition of τ.

Compatibility of A and B as bimodules (and I)

Proof.

We can expand the other diagram as follows:

$$\begin{tikzcd}
B\otimes A\otimes A\otimes A \arrow[r, "1\otimes 1\otimes m_A"] \arrow[dd, "\tau\otimes 1\otimes 1"'] & B\otimes A\otimes A \arrow[r, "1\otimes m_A"] \arrow[d, "\tau\otimes 1"] & B\otimes A \arrow[r, "\tau"] & A\otimes B \\
 & A\otimes B\otimes A \arrow[rr, "1\otimes\tau"] & & A\otimes A\otimes B \arrow[u, "m_A\otimes 1"'] \\
A\otimes B\otimes A\otimes A \arrow[ur, "1\otimes 1\otimes m_A"] \arrow[r, "1\otimes\tau\otimes 1"] & A\otimes A\otimes B\otimes A \arrow[rr, "1\otimes 1\otimes\tau"] & & A\otimes A\otimes A\otimes B \arrow[u, "1\otimes m_A\otimes 1"']
\end{tikzcd}$$

the top right and bottom diagrams are commutative by the above, and the left square is commutative because the functions are acting in terms of the tensor product that do not interfere with each other.
The second part of the statement follows analogously. □

Compatibility maps (I)

Lemma

Then the maps $\tau_{B,\bullet}$ satisfy:

$$\begin{tikzcd}
B \otimes A^{\otimes(n+2)} \arrow[r, "\tau_{B,n}"] \arrow[d, "1\otimes 1_n \otimes m_A"'] \arrow[dr, phantom, "\circlearrowleft"] & A^{\otimes(n+2)} \otimes B \arrow[d, "1_n \otimes m_A \otimes 1"] \\
B \otimes A^{\otimes(n+1)} \arrow[r, "\tau_{B,n-1}"'] & A^{\otimes(n+1)} \otimes B
\end{tikzcd}$$

The maps $\tau_{\bullet,A}$ satisfy the analogous diagram.

Compatibility maps (and II)

Proof.

The result is proven by induction, where the case $n = 0$ has just been done. Consider the hypothesis true for $n-1$, check $\mathbb{N} \ni n \geq 1$:

$$
\begin{tikzcd}
B \otimes A^{\otimes(n+2)} \arrow[rrr, "1\otimes 1_n \otimes m_A"] \arrow[dr, "\tau_{B,n-1}\otimes 1"] \arrow[dd, "\tau_{B,n}"'] & & & B \otimes A^{\otimes(n+1)} \arrow[dl, "\tau_{B,n-2}\otimes 1"'] \arrow[dd, "\tau_{B,n-1}"] \\
& A^{\otimes(n+1)} \otimes B \otimes A \arrow[r, "1_{n-1}\otimes m_A \otimes 1"'] \arrow[dl, "1\otimes\tau"] & A^{\otimes n} \otimes B \otimes A \arrow[dr, "1\otimes\tau"'] & \\
A^{\otimes(n+2)} \otimes B \arrow[rrr, "1_n \otimes m_A \otimes 1"] & & & A^{\otimes(n+1)} \otimes B
\end{tikzcd}
$$

the left and right triangles are commutative by the definition of $\tau_{B,\bullet}$ as a recursion, the top triangle is commutative by the induction hypothesis, and the bottom square commutes because the functions are acting in terms of the tensor product that do not interfere with each other. $\square$

Commutativity with the product in B

Lemma

Let τ be a twisting map for the algebras A and B. Then in the above context we have that for all $n \in \mathbb{N}$:

$$\tau_{B,n} \circ m_B \otimes 1 = 1 \otimes m_B \circ \tau_{B,n} \otimes 1 \circ 1 \otimes \tau_{B,n}.$$

An analogous result follows for $\tau_{\bullet,A}$. To prove this, we will be interpreting it as a diagram.

Proof.

We again use induction. The case $n = -1$ has been proved above. Consider the hypothesis true for $n-1$, we now expand the case $n \in \mathbb{N}$ and obtain commutativity because:

Proof.

$$
\begin{tikzcd}
B \otimes B \otimes A^{\otimes(n+2)} \arrow[rr, "1\otimes\tau_{B,n}"] \arrow[ddrr, "1\otimes\tau_{B,n-1}\otimes 1"] \arrow[dddddddd, "m_B\otimes 1"'] & & B \otimes A^{\otimes(n+2)} \otimes B \arrow[rr, "\tau_{B,n}\otimes 1"] \arrow[ddddrr, "\tau_{B,n-1}\otimes 1"] & & A^{\otimes(n+2)} \otimes B \otimes B \arrow[dddddddd, bend left, "1\otimes m_B"] \\
& & & & \\
& & B \otimes A^{\otimes(n+1)} \otimes B \otimes A \arrow[uu, "1\otimes 1\otimes\tau_{B,A}"'] \arrow[dd, "\tau_{B,n-1}\otimes 1\otimes 1"'] & & \\
& & & & \\
& & A^{\otimes(n+1)} \otimes B \otimes B \otimes A \arrow[rr, "1\otimes 1\otimes\tau_{B,A}"'] \arrow[dd, "1\otimes m_B\otimes 1"'] & & A^{\otimes(n+1)} \otimes B \otimes A \otimes B \arrow[uuuu, "1\otimes\tau_{B,A}\otimes 1"'] \\
& & & & \\
& & A^{\otimes(n+1)} \otimes B \otimes A \arrow[ddrr, "1\otimes\tau_{B,A}"] & & \\
& & & & \\
B \otimes A^{\otimes(n+2)} \arrow[uurr, "\tau_{B,n-1}\otimes 1"'] \arrow[rrrr, "\tau_{B,n}"] & & & & A^{\otimes(n+2)} \otimes B
\end{tikzcd}
$$

the left diagram is the induction hypothesis, the top left, top right and bottom triangles are the definition of $\tau_{B,n}$, the right diagram is the case $n=-1$, and in the central square the functions do not interfere. □

Commutativity with the bimodule structure

Lemma

Let τ be a twisting map for the algebras A and B. Then in the above context we have that for all $n \in \mathbb{N}$:

$$\tau_{B,n} \circ 1 \otimes \rho_{A,n} = \rho_{A,n} \otimes 1 \circ 1 \otimes 1 \otimes \tau \circ 1 \otimes \tau_{B,n} \otimes 1 \circ \tau \otimes 1 \otimes 1.$$

An analogous result follows for $\tau_{\bullet,A}$. Again, we interpret this as a diagram.

Proof.

We again use induction. This time we need the case $n = -1$ as above, and also the case $n = 0$:

Proof.

$$\begin{tikzcd}
B \otimes A \otimes A \otimes A \otimes A \arrow[r, "1\otimes m_A\otimes 1\otimes 1"'] \arrow[d, "\tau\otimes 1\otimes 1\otimes 1"'] & B \otimes A \otimes A \otimes A \arrow[r, "1\otimes 1\otimes m_A"] \arrow[d, "\tau\otimes 1\otimes 1"] & B \otimes A \otimes A \arrow[r, "\tau\otimes 1"] & A \otimes B \otimes A \arrow[d, "1\otimes\tau"] \\
A \otimes B \otimes A \otimes A \otimes A \arrow[d, "1\otimes\tau\otimes 1\otimes 1"'] & A \otimes B \otimes A \otimes A \arrow[urr, "1\otimes 1\otimes m_A"] & & A \otimes A \otimes B \\
A \otimes A \otimes B \otimes A \otimes A \arrow[r, "1\otimes 1\otimes 1\otimes m_A"] \arrow[ur, "m_A\otimes 1\otimes 1\otimes 1"] \arrow[d, "1\otimes 1\otimes\tau\otimes 1"'] & A \otimes A \otimes B \otimes A \arrow[r, "1\otimes 1\otimes\tau"] \arrow[uurr, "m_A\otimes 1\otimes 1"] & A \otimes A \otimes A \otimes B \arrow[ur, "m_A\otimes 1\otimes 1"] & \\
A \otimes A \otimes A \otimes B \otimes A \arrow[r, "1\otimes 1\otimes 1\otimes\tau"'] & A \otimes A \otimes A \otimes A \otimes B \arrow[r, "m_A\otimes 1\otimes 1\otimes 1"'] \arrow[u, "1\otimes 1\otimes m_A\otimes 1"] & A \otimes A \otimes A \otimes B \arrow[uur, "1\otimes m_A\otimes 1"'] &
\end{tikzcd}$$

the top left and bottom left diagrams are commutative as seen above, and the remaining squares are commutative because the functions are acting in terms of the tensor product that do not interfere with each other.

A scary diagram can be used to finish induction. □

Lifting to a chain map

Lemma

Let τ be a twisting map for the algebras A and B. Then in the above context we have that for all $n \in \mathbb{N}$:

$$\tau_{B,n+1} \circ 1 \otimes d_n = d_n \otimes 1 \circ \tau_{B,n+2}.$$

An analogous result follows for $\tau_{\bullet,A}$. As usual, we interpret it as a diagram.

Proof.

We again use induction. The case $n = 0$ has been proved above. Consider the hypothesis true for $n-1$, we check $n \in \mathbb{N}$:

$$\begin{tikzcd} B \otimes A^{\otimes(n+2)} \arrow[r, "1\otimes d_n"] \arrow[d, "\tau_{B,n}"'] & B \otimes A^{\otimes(n+1)} \arrow[d, "\tau_{B,n-1}"] \\ A^{\otimes(n+2)} \otimes B \arrow[r, "d_n\otimes 1"] & A^{\otimes(n+1)} \otimes B \end{tikzcd}$$

Proof.

Now $1 \otimes d_n = 1 \otimes m_A \otimes 1 - 1 \otimes 1 \otimes d_{n-1}$, $d_n \otimes 1 = m_A \otimes 1 - 1 \otimes d_{n-1} \otimes 1$ so we can express the previous diagram as the sum of the following two:

$$
\begin{tikzcd}
B \otimes A^{\otimes(n+2)} \arrow[rrr, "1\otimes m_A\otimes 1_n"] \arrow[dd, "\tau_{B,n}"'] \arrow[dr, "\tau_{B,n-1}\otimes 1"] & & & B \otimes A^{\otimes(n+1)} \arrow[dl, "\tau_{B,n-2}\otimes 1"'] \arrow[dd, "\tau_{B,n-1}"] \\
& A^{\otimes(n+1)} \otimes B \otimes A \arrow[r, "m_A\otimes 1"'] \arrow[dl, "1\otimes\tau"] & A^{\otimes n} \otimes B \otimes A \arrow[dr, "1\otimes\tau"'] & \\
A^{\otimes(n+2)} \otimes B \arrow[rrr, "m_A\otimes 1"] & & & A^{\otimes(n+1)} \otimes B
\end{tikzcd}
$$

$$
\begin{tikzcd}
B \otimes A^{\otimes(n+2)} \arrow[rrr, "\pm 1\otimes 1\otimes d_{n-1}"] \arrow[dd, "\tau_{B,n}"'] \arrow[dr, "\tau\otimes 1"] & & & B \otimes A^{\otimes(n+1)} \arrow[dl, "\tau\otimes 1"'] \arrow[dd, "\tau_{B,n-1}"] \\
& A^{\otimes(n-1)} \otimes B \otimes A \arrow[r, "\pm 1\otimes 1\otimes d_{n-1}"'] \arrow[dl, "1\otimes\tau_{B,n-1}"] & A^{\otimes(n-2)} \otimes B \otimes A \arrow[dr, "1\otimes\tau_{B,n-2}"'] & \\
A^{\otimes(n+2)} \otimes B \arrow[rrr, "\pm 1\otimes d_{n-1}\otimes 1"] & & & A^{\otimes(n+1)} \otimes B
\end{tikzcd}
$$

Hochschild Cohomology

Here having k to be a field does matter.

Definition

The *Hochschild cohomology* of a k algebra A with coefficients in a left A^e module M is $HH^\bullet(A, M) = \bigoplus_{n\in\mathbb{N}} HH^n(A, M)$, where for $n \in \mathbb{N}$:

$$HH^n(A, M) = \mathrm{Ext}^n_{A^e}(A, M).$$

Hence to compute the Hochschild cohomology we need A^e projective resolutions of A. We now provide a canonical one.

Hochschild cohomology of the tensor product

It is possible (under some finiteness assumptions) to understand the (co)homology theory of a tensor product in terms of the (co)homology of the original factors:

Theorem (Le-Zhou)

There is an isomorphism of Gerstenhaber algebras:

$$HH^*(A \otimes B) \cong HH^*(A) \otimes HH^*(B).$$

Hochschild cohomology of the twisted tensor product

Does it make sense (i.e. is it defined) to take $HH^*(A) \otimes_\tau HH^*(B)$? Under which hypothesis?

Question

Is there an isomorphism:

$$HH^*(A \otimes_\tau B) \cong HH^*(A) \otimes_\tau HH^*(B)?$$

As (graded) k modules? As (graded) algebras? As Gerstenhaber algebras?

Something to take home

- Visualization of equations through diagrams enable sound logical reasoning and lets us understand what is happening.
- Hochschild cohomology of individual algebras can be used to obtain Hochschild cohomology of tensor products of algebras.
- In non commutative algebra the usual tensor product takes the form of a twisted tensor product. Understanding it is useful.

Thank you!