Using Relative Homological Algebra in Hochschild Cohomology

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Outline

Why do we care?

- Homology is a **useful tool** in studying algebraic objects: it provides insight into their properties and structure(s).
- When algebraic objects appear in other fields (representation theory, geometry, topology...), homology **encodes meaningful information** to that field (semisimplicity, genus, path-connectedness...).
- Hochschild cohomology encodes information on **deformations**, **smoothness** and **representations** of algebras, among others.

Relative exact sequences

Definition

Let R be a ring, a sequence of R modules:

$$\cdots \longrightarrow C_i \xrightarrow{t_i} C_{i-1} \longrightarrow \cdots$$

is called *exact* if $\mathrm{Im}(t_i) = \mathrm{Ker}(t_{i-1})$.

Definition

Let $1_R \in S \subseteq R$ a subring. An exact sequence of R modules is called (R, S) *exact* if $\mathrm{Ker}(t_i)$ is a direct summand of C_i as an S module.

This is equivalent to the sequence splitting, and to the existence of an S homotopy (hence the sequence is exact as S modules).

Relative projective modules (I)

Definition

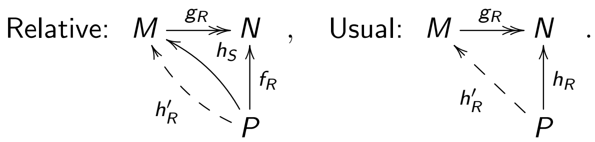
An R module P is said to be (R, S) *projective* if, for every (R, S) exact sequence $M \xrightarrow{g} N \longrightarrow 0$ and every R homomorphism $h : P \longrightarrow N$, there is an R homomorphism $h' : P \longrightarrow M$ with $gh' = h$.

Relative: $\begin{array}{ccccc} M & \underset{f_S}{\overset{g_R}{\rightleftarrows}} & N & \longrightarrow & 0 \\ & \nwarrow^{h'_R} & \uparrow h_R & & \\ & & P & & \end{array}$, Usual: $\begin{array}{ccccc} M & \xrightarrow{g_R} & N & \longrightarrow & 0 \\ & \nwarrow^{h'_R} & \uparrow h_R & & \\ & & P & & \end{array}$.

Relative projective modules (and II)

Definition

An R module P is said to be (R, S) *projective* if, for every exact sequence $M \xrightarrow{g} N \longrightarrow 0$ of R modules, every R homomorphism $f : P \longrightarrow N$ and every S homomorphism $h : P \longrightarrow M$ with $gh = f$, there is an R homomorphism $h' : P \longrightarrow M$ with $gh' = f$.



Some lifting properties

Lemma

For every S module N, the R module $R \otimes_S N$ is (R, S) projective.

Proposition

Let V be an (R, S) projective R module, $f : M \longrightarrow N$ a homomorphism of right R modules with:

$$M \otimes_S V \xhookrightarrow{f \otimes 1_V} N \otimes_S V \ , \quad \text{then} \quad M \otimes_R V \xhookrightarrow{f \otimes 1_V} N \otimes_R V \ .$$

Relative Comparison Theorem

Theorem

Let M and N be R modules and two chain complexes (that is, the composition of consecutive homomorphisms is zero):

$$P: \quad \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$
$$Q: \quad \cdots \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow N \longrightarrow 0$$

such that P_i is (R,S) projective for all $i \in \mathbb{N}$ and $Q_\bullet$ is (R,S) exact. If $f: M \longrightarrow N$ is an R homomorphism then there exists a chain map $f_\bullet : P_\bullet \longrightarrow Q_\bullet$ lifting it, that is, the following diagram is commutative:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle f_1} & & \Big\downarrow{\scriptstyle f_0} & & \Big\downarrow{\scriptstyle f} & & \\ \cdots & \longrightarrow & Q_1 & \longrightarrow & Q_0 & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

Relative Ext

Let M and N be R modules and:

$$P: \quad \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \longrightarrow 0$$

an (R,S) exact sequence where P_i is (R,S) projective for all $i \in \mathbb{N}$ (that is an (R,S) projective resolution). Consider the complex $\mathrm{Hom}_R(P_\bullet, N)$:

$$0 \longrightarrow \mathrm{Hom}_R(M,N) \xrightarrow{d_0^*} \mathrm{Hom}_R(P_0,N) \xrightarrow{d_1^*} \mathrm{Hom}_R(P_1,N) \xrightarrow{d_2^*} \cdots$$

Definition

We define:

$$\mathrm{Ext}^0_{(R,S)}(M,N) = \mathrm{Ker}(d_1^*),$$

$$\mathrm{Ext}^n_{(R,S)}(M,N) = \mathrm{Ker}(d_{n+1}^*)/\mathrm{Im}(d_n^*) \text{ for } n \geq 1.$$

Analogous results and recovery of Homological Algebra

In virtue of the Comparison Theorem:

- The $\mathrm{Ext}^n_{(R,S)}(M,N)$ for $n \in \mathbb{N}$ are independent of the resolution.
- A pair of R homomorphisms $f : M \longrightarrow M'$, $g : N \longrightarrow N'$ induce a unique $\phi_{f,g} : \mathrm{Ext}^n_{(R,S)}(M',N) \longrightarrow \mathrm{Ext}^n_{(R,S)}(M,N')$ and functoriality.

Remark

- If S is semisimple, meaning that every R exact sequence is (R,S) exact, or
- If R is projective as an S module, then:

$$\mathrm{Ext}^n_{(R,S)}(M,N) = \mathrm{Ext}^n_R(M,N) \text{ for } n \in \mathbb{N}.$$

Algebras over a ring (I)

Definition

Let k be an associative commutative ring. We say that A is a k *algebra* if it is a k module and a ring, where the product $\mu : A \times A \longrightarrow A$ is bilinear.

Examples:

- $k[x]$.
- $k[x_1, \ldots, x_n]$.
- $k[x]/(x^n)$ for $n \in \mathbb{N}$.

There are noncommutative algebras.

Algebras over a ring (II)

Definition

Let A be a k algebra. We define A^{op} the *opposite algebra of* A as the vector space A with multiplication $\mu_{op} : A \times A \longrightarrow A$ given by:

$$\mu_{op}(a, b) = \mu(b, a) \text{ for all } a, b \in A.$$

Examples:

- $k[x]^{op} = k[y]$.
- $k[x_1, \ldots, x_n]^{op} = k[y_1, \ldots, y_n]$.
- $k[x]/(x^n)^{op} = k[y]/(y^n)$ for $n \in \mathbb{N}$.

Since they are commutative.

Algebras over a ring (and III)

Definition

Let A be a k algebra. We define A^e the *enveloping algebra of* A as the vector space $A \otimes A^{op}$ with multiplication $\mu^e : A^e \times A^e \longrightarrow A^e$ given by:

$$\mu^e((a_1 \otimes b_1),(a_2 \otimes b_2)) = \mu(a_1,a_2) \otimes \mu_{op}(b_1,b_2) = a_1 a_2 \otimes b_2 b_1$$

for all $a_1, a_2, b_1, b_2 \in A$.

Examples:

- $k[x]^e = k[x] \otimes k[y] \cong k[x,y]$.
- $k[x_1,\dots,x_n]^e = k[x_1,\dots,x_n] \otimes k[y_1,\dots,y_n] \cong k[x_1,\dots,x_n,y_1,\dots,y_n]$.
- $k[x]/(x^n)^e = k[y]/(y^n) \otimes k[y]/(y^n) \cong k[x,y]/(x^n,y^n)$ for $n \in \mathbb{N}$.

Modules and bimodules over an algebra

Remark

There is a one to one correspondence between the bimodules M over a k algebra A and the (right or left) modules M over A^e.

Note that A is a left A^e module under:

$$(a \otimes b) \cdot c = acb \text{ for all } a, b, c \in A.$$

More generally, $A^{\otimes n} = A \otimes \overset{(n)}{\cdots} \otimes A$ is is a left A^e module under:

$$(a \otimes b) \cdot (c_1 \otimes c_2 \cdots \otimes c_{n-1} \otimes c_n) = ac_1 \otimes c_2 \cdots \otimes c_{n-1} \otimes c_n b$$

for all $a, b, c_1, \ldots, c_n \in A$.

The Bar sequence

Consider the sequence of left A^e modules:

$$\cdots \xrightarrow{d_3} A^{\otimes 4} \xrightarrow{d_2} A^{\otimes 3} \xrightarrow{d_1} A \otimes A \xrightarrow{\mu} A \longrightarrow 0$$

with:

$$d_n(a_0 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}$$

for all $a_0, \ldots, a_{n+1} \in A$. This is a complex.

The Bar resolution

The bar sequence has a contracting homotopy $s_n : A^{\otimes(n+2)} \longrightarrow A^{\otimes(n+3)}$:

$$s_n(a_0 \otimes \cdots \otimes a_{n+1}) = 1 \otimes a_0 \otimes \cdots \otimes a_{n+1}$$

for all $a_0, \ldots, a_{n+1} \in A$.

Definition

Let A be a k algebra. We define the *bar complex of* A as the truncated complex:

$$B(A): \quad \cdots \xrightarrow{d_3} A^{\otimes 4} \xrightarrow{d_2} A^{\otimes 3} \xrightarrow{d_1} A \otimes A \longrightarrow 0$$

and write $B_n(A) = A^{\otimes(n+2)}$ for $n \in \mathbb{N}$.

Hochschild Cohomology

Let M a left A^e module, consider the complex $\mathrm{Hom}_{A^e}(B(A), M)$:

$$0 \longrightarrow \mathrm{Hom}_{A^e}(A \otimes A, M) \xrightarrow{d_1^*} \mathrm{Hom}_{A^e}(A^{\otimes 3}, M) \xrightarrow{d_2^*} \mathrm{Hom}_{A^e}(A^{\otimes 4}, M) \xrightarrow{d_3^*} \cdots$$

Definition

The *Hochschild cohomology* of A with coefficients in a left A^e module M is the cohomology of $\mathrm{Hom}_{A^e}(B(A), M)$, equivalently:

$$HH^n(A, M) = H^n(\mathrm{Hom}_{A^e}(A^{\otimes \bullet}, M)) = \mathrm{Ker}(d_{n+1}^*)/\mathrm{Im}(d_n^*)$$

for $n \in \mathbb{N}$.

This construction reminds of derived functors, particularly Ext.

Hochschild Cohomology from Relative Homological Algebra

Theorem

Let M be a left A^e module and consider $k \subset A^e$ as a subring. Then:

$$HH^n(A, M) = \mathrm{Ext}^n_{(A^e,k)}(A, M) \text{ for } n \in \mathbb{N}.$$

In particular when k is a field, $HH^\bullet$ is $\mathrm{Ext}^\bullet$.

Ongoing research

- How good is *relative* Hochschild Cohomology?

$$HH^n_{(A,B)}(A, M) = \mathrm{Ext}^n_{(A^e, B\otimes A^{op})}(A, M)$$

- Does it have a Gerstenhaber *bracket*?

$$[f, g] = f \circ g - (-1)^{(m-1)(n-1)} g \circ f$$

Something to take home

- Hochschild cohomology is intimately related with the Ext functor.
- Over a field this is well behaved and fairly well understood.
- There is a lot of progress to be made in natural generalizations.

Thank you!

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