Hochschild cohomology of twisted tensor product algebras (and brackets for certain quantum complete intersections)

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Outline

Why do we care?

1. Sometimes we can understand the (co)homology theory of a tensor product in terms of the (co)homology of the original factors.
2. This understanding relies on the tensor product of projective resolutions for the factor algebras being a projective resolution for the tensor product of the algebras.
3. Čap, Schichl, and Vanžura introduced twisted tensor products in 1995 as an analogue for non commutative algebras.
4. In concrete settings, a construction similar to the commutative case have been achieved, yielding similar results.
5. Shepler and Witherspoon unified many of these constructions in 2018.

Computational use in concrete settings

1. Negron and Witherspoon in 2016 develop techniques to construct Gerstenhaber brackets on Hochschild cohomology.
2. Grimley, Nguyen, and Witherspoon augmented these techniques in 2017, constructing and computing the Gerstenhaber bracket in some twisted tensor products.
3. Can these conditions be relaxed to compute the Gerstenhaber bracket of a twisted tensor product? If so, how much?

Algebras over a ring (I)

Definition

Let k be an associative commutative ring. We say that A is a k *algebra* if it is a k module and a ring, where the product $\mu : A \times A \longrightarrow A$ is bilinear.

Examples:

- Commutative: $k[x]$, $k[x_1, \ldots, x_n]$, $k[x]/(x^n)$ for $n \in \mathbb{N}$.
- Noncommutative: $k\langle x, y\rangle/(yx - xy - x^2)$.

Definition

Let A be a k algebra. We define A^{op} the *opposite algebra of* A as the vector space A with multiplication $\mu_{op} : A \times A \longrightarrow A$ given by:

$$\mu_{op}(a, b) = \mu(b, a) \text{ for all } a, b \in A.$$

Algebras over a ring (and II)

Definition

Let A be a k algebra. We define A^e the *enveloping algebra of* A as the vector space $A \otimes A^{op}$ with multiplication $\mu^e : A^e \times A^e \longrightarrow A^e$ given by:

$$\mu^e((a_1 \otimes b_1), (a_2 \otimes b_2)) = \mu(a_1, a_2) \otimes \mu_{op}(b_1, b_2) = a_1 a_2 \otimes b_2 b_1$$

for all $a_1, a_2, b_1, b_2 \in A$.

Examples:

- $k[x]^e = k[x] \otimes k[y] \cong k[x, y]$.
- $k[x]/(x^n)^e = k[x]/(x^n) \otimes k[y]/(y^n) \cong k[x, y]/(x^n, y^n)$ for $n \in \mathbb{N}$.

Modules and bimodules over an algebra

Remark

There is a one to one correspondence between the bimodules M over a k algebra A and the (right or left) modules M over A^e.

Note that A is a left A^e module under:

$$(a \otimes b) \cdot c = acb \text{ for all } a, b, c \in A.$$

More generally, $A^{\otimes n} = A \otimes \overset{(n)}{\cdots} \otimes A$ is a left A^e module under:

$$(a \otimes b) \cdot (c_1 \otimes c_2 \otimes \cdots \otimes c_{n-1} \otimes c_n) = ac_1 \otimes c_2 \otimes \cdots c_{n-1} \otimes c_n b$$

for all $a, b, c_1, \ldots, c_n \in A$.

The Bar sequence

Consider the sequence of left A^e modules:

$$\cdots \xrightarrow{d_3} A^{\otimes 4} \xrightarrow{d_2} A^{\otimes 3} \xrightarrow{d_1} A \otimes A \xrightarrow{\mu} A \longrightarrow 0$$

with:

$$d_n(a_0 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}$$

for all $a_0, \cdots, a_{n+1} \in A$. This is a complex.

The Bar resolution

The bar sequence has a contracting homotopy $s_n : A^{\otimes(n+2)} \longrightarrow A^{\otimes(n+3)}$:

$$s_n(a_0 \otimes \cdots \otimes a_{n+1}) = 1 \otimes a_0 \otimes \cdots \otimes a_{n+1}$$

for all $a_0, \cdots, a_{n+1} \in A$.

Definition

Let A be a k algebra. We define the *bar complex of A* as the truncated complex:

$$\mathbb{B}(A) : \quad \cdots \xrightarrow{d_3} A^{\otimes 4} \xrightarrow{d_2} A^{\otimes 3} \xrightarrow{d_1} A \otimes A \longrightarrow 0$$

and write $\mathbb{B}_n(A) = A^{\otimes(n+2)}$ for $n \in \mathbb{N}$.

Hochschild cohomology (I)

Let M be a left A^e module, consider the complex $\mathrm{Hom}_{A^e}(\mathbb{B}(A), M)$:

$$0 \longrightarrow \mathrm{Hom}_{A^e}(A \otimes A, M) \xrightarrow{d_1^*} \mathrm{Hom}_{A^e}(A^{\otimes 3}, M) \xrightarrow{d_2^*} \mathrm{Hom}_{A^e}(A^{\otimes 4}, M) \xrightarrow{d_3^*} \cdots$$

Definition

The *Hochschild cohomology* of A with coefficients in a left A^e module M is the cohomology of $\mathrm{Hom}_{A^e}(\mathbb{B}(A), M)$, equivalently:

$$HH^n(A, M) = H^n(\mathrm{Hom}_{A^e}(A^{\otimes \bullet}, M)) = \mathrm{Ker}(d_{n+1}^*)/\mathrm{Im}(d_n^*)$$

for $n \in \mathbb{N}$.

This construction reminds of derived functors, particularly Ext.

Hochschild cohomology (and II)

Theorem

Let M be a left A^e module and consider $k \subset A^e$ as a subring. Then:

$$HH^n(A, M) = \mathrm{Ext}^n_{(A^e,k)}(A, M) \text{ for } n \in \mathbb{N}.$$

In particular when k is a field, $HH^\bullet$ is $\mathrm{Ext}^\bullet$. In this case $A^{\otimes n} \cong \bigoplus_{i \in I} k\alpha_i$ as k modules:

$$A^{\otimes(n+2)} \cong A^e \otimes A^{\otimes n} \cong \bigoplus_{i \in I} A^e(1 \otimes 1 \otimes \alpha_i)$$

so $A^{\otimes(n+2)}$ are free A^e modules, and the complex is a free resolution. For this and other technical reasons, from now on we take k to be a field.

Twisted tensor product algebra

Definition

Let A, B two algebras over k. We say that a bijective k linear map $\tau : B \otimes A \longrightarrow A \otimes B$ is a *twisting map* if $\tau(1_B \otimes a) = a \otimes 1_B$ and $\tau(b \otimes 1_A) = 1_A \otimes b$ for all $a \in A$, $b \in B$ and:

$$\begin{array}{ccccc} B \otimes B \otimes A \otimes A & \xrightarrow{m_B \otimes m_A} & B \otimes A & \xrightarrow{\tau} & A \otimes B \\ \Big\downarrow{\scriptstyle 1 \otimes \tau \otimes 1} & & \circlearrowleft & & \Big\uparrow{\scriptstyle m_A \otimes m_B} \\ B \otimes A \otimes B \otimes A & \xrightarrow{\tau \otimes \tau} & A \otimes B \otimes A \otimes B & \xrightarrow{1 \otimes \tau \otimes 1} & A \otimes A \otimes B \otimes B \end{array}$$

Definition

Under this condition, the *twisted tensor product algebra* $A \otimes_\tau B$ is the vector space $A \otimes B$ with multiplication:

$$m_\tau : (A \otimes B) \otimes (A \otimes B) \xrightarrow{1 \otimes \tau \otimes 1} A \otimes A \otimes B \otimes B \xrightarrow{m_A \otimes m_B} A \otimes B$$

Bimodule compatible with the twisting (I)

Definition

We say that an A bimodule M, whose bimodule structure is given by $\rho_A : A \otimes M \otimes A \longrightarrow M$, is *compatible with* τ if there exist a bijective k linear map $\tau_{B,M} : B \otimes M \longrightarrow M \otimes B$ such that:

1. $\tau_{B,M}$ is well behaved with respect to the algebra structure of B,
2. the module structure of M is well behaved (via $\tau_{B,M}$) with respect to the algebra structure of B and the twisting map τ.

We analogously define how a B bimodule N is compatible with τ via $\tau_{N,A}$.

Bimodule compatible with the twisting (and II)

$$
\begin{array}{ccccc}
B \otimes B \otimes M & \xrightarrow{1 \otimes \tau_{B,M}} & B \otimes M \otimes B & \xrightarrow{\tau_{B,M} \otimes 1} & M \otimes B \otimes B \\
\downarrow{\scriptstyle m_B \otimes 1} & & & & \downarrow{\scriptstyle 1 \otimes m_B} \\
B \otimes M & & \xrightarrow{\qquad \tau_{B,M} \qquad} & & M \otimes B \\
\uparrow{\scriptstyle 1 \otimes \rho_A} & & & & \uparrow{\scriptstyle \rho_A \otimes 1} \\
B \otimes A \otimes M \otimes A & & & & A \otimes M \otimes A \otimes B \\
\downarrow{\scriptstyle \tau \otimes 1 \otimes 1} & & & & \uparrow{\scriptstyle 1 \otimes 1 \otimes \tau} \\
A \otimes B \otimes M \otimes A & & \xrightarrow{\quad 1 \otimes \tau_{B,M} \otimes 1 \quad} & & A \otimes M \otimes B \otimes A
\end{array}
$$

If M and N are A and B bimodules via ρ_A and ρ_B compatible with τ via $\tau_{B,M}$ and $\tau_{N,A}$ respectively, then:

$$\begin{array}{ccc} (A \otimes_\tau B) \otimes (M \otimes N) \otimes (A \otimes_\tau B) & \xrightarrow{\rho_{A\otimes_\tau B}} & M \otimes N \\ \Big\downarrow {\scriptstyle 1\otimes\tau_{B,M}\otimes\tau_{N,A}\otimes 1} & \circlearrowleft & \Big\uparrow {\scriptstyle \rho_A\otimes\rho_B} \\ A \otimes M \otimes B \otimes A \otimes N \otimes B & \xrightarrow{1\otimes 1\otimes\tau\otimes 1\otimes 1} & A \otimes M \otimes A \otimes B \otimes N \otimes B \end{array}$$

defines a natural structure of $A \otimes_\tau B$ bimodule over $M \otimes N$ via $\rho_{A\otimes_\tau B}$.

Compatibility of resolutions (I)

Let $P_\bullet(M)$ be an A^e projective resolution of M and $P_\bullet(N)$ a B^e projective resolution of N:

$$\cdots \longrightarrow P_2(M) \longrightarrow P_1(M) \longrightarrow P_0(M) \longrightarrow M \longrightarrow 0,$$
$$\cdots \longrightarrow P_2(N) \longrightarrow P_1(N) \longrightarrow P_0(N) \longrightarrow N \longrightarrow 0.$$

Consider the complexes $P_\bullet(N) \otimes A$, $A \otimes P_\bullet(N)$, $P_\bullet(M) \otimes B$, $B \otimes P_\bullet(M)$. As exact sequences of vector spaces any k linear maps:

$$\tau_{N,A} : N \otimes A \longrightarrow A \otimes N \quad \text{and} \quad \tau_{B,M} : B \otimes M \longrightarrow M \otimes B$$

can be lifted to k linear chain maps:

$$\tau_{P_\bullet(N),A} : P_\bullet(N) \otimes A \longrightarrow A \otimes P_\bullet(N), \quad \tau_{B,P_\bullet(M)} : B \otimes P_\bullet(M) \longrightarrow P_\bullet(M) \otimes B,$$

denoted by $\tau_{i,A} := \tau_{P_i(N),A}$ and $\tau_{B,i} := \tau_{B,P_i(M)}$.

Compatibility of resolutions (and II)

Definition

Given M an A bimodule that is compatible with τ, we say that a projective A^e resolution $P_\bullet(M)$ is *compatible with* τ if each $P_i(M)$ is compatible with τ via a map $\tau_{B,i} : B \otimes P_i(M) \longrightarrow P_i(M) \otimes B$ such that $\tau_{B,\bullet}$ is a chain map lifting $\tau_{B,M}$.

Given N a B bimodule compatible with τ, we can analogously define how a projective B^e resolution $P_\bullet(N)$ is compatible with τ via $\tau_{\bullet,A}$.

The bar resolution is compatible with the twisting

Proposition

Let τ be a twisting map for the algebras A and B. Then $\mathbb{B}(A)$ and $\mathbb{B}(B)$, the bar resolutions of A and B respectively, are compatible with τ.

We need to say via which maps.

Definition

For each $n \in \mathbb{N}$ define the maps $\tau_{B,n} : B \otimes \mathbb{B}_n(A) \longrightarrow \mathbb{B}_n(A) \otimes B$ recursively: $\tau_{B,0} := 1 \otimes \tau \circ \tau \otimes 1$, $\tau_{B,n} := 1 \otimes \tau \circ \tau_{B,n-1} \otimes 1$.

Notice that equivalently $\tau_{B,n}$ satisfies:

$$\tau_{B,0} := 1 \otimes \tau \circ \tau \otimes 1, \quad \tau_{B,n} := 1 \otimes \tau_{B,n-1} \circ \tau \otimes 1.$$

We define analogously $\tau_{n,A}$.

Proof.

Both A and B satisfy the prerequisites of compatibility necessary to ask whether $\mathbb{B}(A)$ and $\mathbb{B}(B)$ may be compatible with τ.

To see that $\mathbb{B}(A)$ is compatible with τ we need that for all $n \in \mathbb{N}$:

1. Commutativity with the product in B:

$$\tau_{B,n} \circ m_B \otimes 1 = 1 \otimes m_B \circ \tau_{B,n} \otimes 1 \circ 1 \otimes \tau_{B,n}.$$

2. Commutativity with the bimodule structure:

$$\tau_{B,n} \circ 1 \otimes \rho_{A,n} = \rho_{A,n} \otimes 1 \circ 1 \otimes 1 \otimes \tau \circ 1 \otimes \tau_{B,n} \otimes 1 \circ \tau \otimes 1 \otimes 1.$$

3. Lifting to a chain map:

$$\tau_{B,n+1} \circ 1 \otimes d_n = d_n \otimes 1 \circ \tau_{B,n+2}.$$

The second part of the statement follows analogously. □

Technical requirements

Lemma

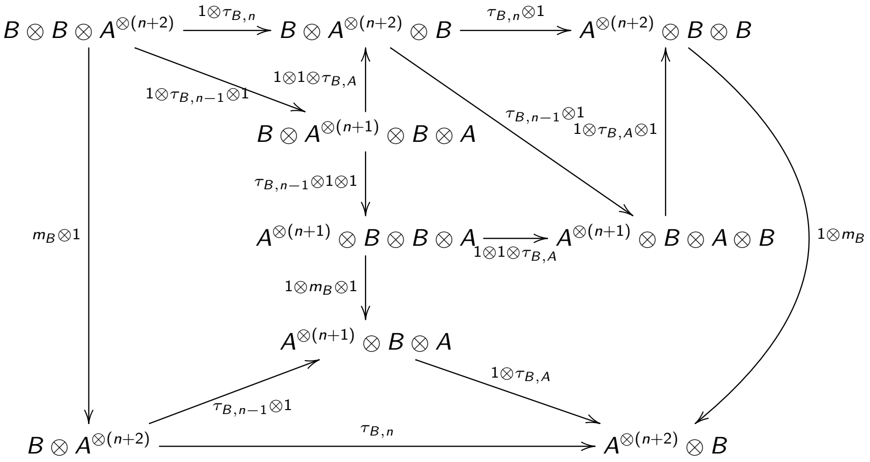
Let τ be a twisting map for the algebras A and B. Then A and B, seen as an A^e module and a B^e module respectively, are compatible with τ via τ.

Lemma

Then the maps $\tau_{B,\bullet}$ satisfy:

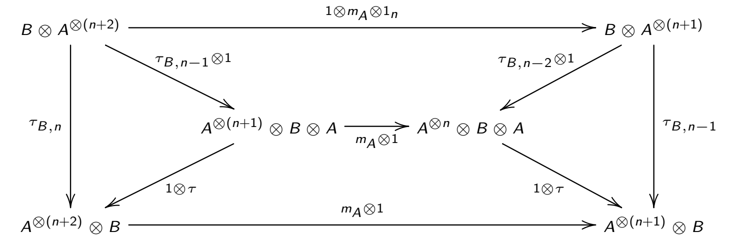
$$\begin{array}{ccc}
B \otimes A^{\otimes(n+2)} & \xrightarrow{\ \tau_{B,n}\ } & A^{\otimes(n+2)} \otimes B \\
{\scriptstyle 1\otimes 1_n \otimes m_A}\downarrow & \circlearrowleft & \downarrow{\scriptstyle 1_n\otimes m_A \otimes 1} \\
B \otimes A^{\otimes(n+1)} & \xrightarrow[\ \tau_{B,n-1}\ ]{} & A^{\otimes(n+1)} \otimes B
\end{array}$$

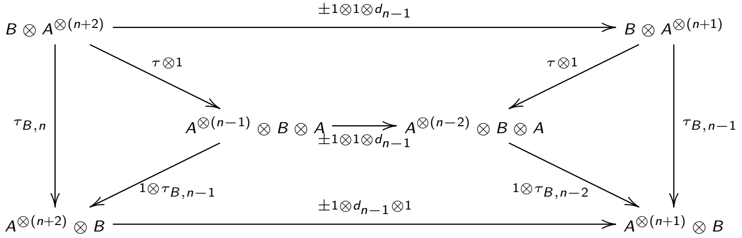
The maps $\tau_{\bullet,A}$ satisfy the analogous diagram.



Uses induction, the definition of $\tau_{B,n}$, and that in some subdiagrams the functions do not interfere.

Since $d_n = m_A \otimes 1 - 1 \otimes d_{n-1}$ we can decompose some diagrams in a sum:





Computing the Gerstenhaber bracket (I)

The Hochschild cohomology of a k algebra A with coefficients in A:

$$HH^\bullet(A,A) = \bigoplus_{n\in\mathbb{N}} HH^n(A,M) = \bigoplus_{n\in\mathbb{N}} \mathrm{Ext}^n_{A^e}(A,M)$$

has an associative graded commutative *cup product*:

$$\smile : HH^m(A,A) \times HH^n(A,A) \longrightarrow HH^{m+n}(A,A)$$

intimately related with a *Gerstenhaber bracket*:

$$[-,-] : HH^m(A,A) \times HH^n(A,A) \longrightarrow HH^{m+n-1}(A,A)$$

making $HH^\bullet(A,A)$ into a graded Lie algebra.

Remark

Computing this bracket provides insightful information about A, albeit being hard. Techniques for doing so are hence useful.

Computing the Gerstenhaber bracket (II)

Grimley, Nguyen, and Witherspoon treated the case of twists arising from:

$$t : A \otimes_{\mathbb{Z}} B \longrightarrow k^{\times}$$

a homomorphism of abelian groups, where the twisted tensor product $R \otimes^t S$ has multiplication induced by t on the middle elements of the k vector space $R \otimes S \otimes R \otimes S$.

They computed brackets for the quantum complete intersections:

$$k\langle x, y\rangle/(x^2, y^2, xy + qyx) \text{ for some } q \in k^{\times}.$$

Computing the Gerstenhaber bracket (and III)

Computational applications:

We extended results from Grimley, Nguyen, and Witherspoon, showing how to compute it from some compatible resolutions.

We then applied some of these techniques to compute the bracket for:

$$k\langle x, y\rangle/(xy - yx - y^2),$$

the Jordan plane.

Isomorphisms of Hochschild cohomology (I)

It is possible (under some finiteness assumptions) to understand the (co)homology theory of a tensor product in terms of the (co)homology of the original factors:

Theorem (Le-Zhou)

There is an isomorphism of Gerstenhaber algebras:

$$HH^*(A \otimes B) \cong HH^*(A) \otimes HH^*(B).$$

Grimley, Nguyen, and Witherspoon generalized that to:

$$HH^{*,A' \oplus B'}(R \otimes_k^t S) \cong HH^{*,A'}(R) \otimes HH^{*,B'}(B).$$

Our methods re-prove the result by Le-Zhou using more transparent techniques.

Isomorphisms of Hochschild cohomology (and I)

It may be possible to use our methods to generalize those results. Does it make sense (i.e. is it defined) to take $HH^*(A) \otimes_\tau HH^*(B)$? Under which hypothesis?

Question

Is there an isomorphism:

$$HH^*(A \otimes_\tau B) \cong HH^*(A) \otimes_\tau HH^*(B)?$$

As (graded) k modules? As (graded) algebras? As Gerstenhaber algebras?

Something to take home

- Visualization of equations through diagrams enable sound logical reasoning and lets us understand what is happening.
- Hochschild cohomology of individual algebras can be used to obtain Hochschild cohomology of tensor products of algebras.
- In non commutative algebra the usual tensor product takes the form of a twisted tensor product. Understanding it is useful.

Thank you!