Hochschild Cohomology of Twisted Tensor Product Algebras

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March 30, 2019

Hochschild cohomology, as all cohomologies do, encode information about the algebra. We will be working with algebras over a field, so in this case the Hochschild cohomology is essentially the Ext functor. The original way to compute this cohomology employed the so called "bar resolution", which is a bunch of copies of the algebra tensored together. We look at both this and the slightly smaller Koszul resolution.

Given two algebras, their twisted tensor product is the non-commutative analogue of the usual tensor product, via a map called twisting. Our goal is to understand the Hochschild cohomology of $A \otimes_\tau B$ in terms of the Hochschild cohomology of A and the Hochschild cohomology of B; in particular given resolutions of A and B, we work towards obtaining a resolution of $A \otimes_\tau B$. For this, we use the notion of a resolution being compatible with the twisting, that is, a resolution satisfying the natural conditions saying that the structure of both the modules and the maps in the resolution play well with the twisting. These conditions can be thought of as analogues to, for example, bilinearity in the usual case of a tensor product.

The tools that we use are basically fancy diagram chasing: by the nature of the bar and Koszul resolutions, we can reduce the compatibility conditions of the resolutions to checking that some diagrams commute. These diagrams are mostly formed by successive applications of the twisting, accompanied by the algebra multiplication and some module structures. These diagrams may look fairly innocent, but after a few juggling they look something like what you can see in the screen, where notice that what we are essentially doing is switching the order of A and B. However, since this switch is provided with a structure inherited from the twisting, we need to be careful about what is compatible, and thus commutes, and what isn't, and thus doesn't.

While a lot of work still needs to be done, understanding this will give us insight into how some geometric objects behave. For example, progress has been made in the following quantum complete intersections, where the structure of the Hochschild cohomology is better understood, enabling their geometric understanding.