THE RELATIVE KÜNNETH THEOREM

OR THE SEARCH FOR A RELATIVE SUPPORT.

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June 15, 2022.

REPRESENTATIONS OF A FINITE GROUP

(1)

(when k has characteristic zero)

Maschke's Theorem: kG is semisimple.

Artin-Wedderburn Theorem: Semisimple rings are isomorphic to a product of finitely many matrix rings over division rings.

REPRESENTATIONS OF A FINITE GROUP

(2)

(when k has positive characteristic dividing the order of G)

kG is not semisimple: for each ideal I there is no left ideal J with $kG = I \oplus J$.

However, we can measure the failure of semisimplicity using the $\boxed{\text{stable category.}}$

3

THE STABLE MODULE CATEGORY

Our hopes of understanding modkG are slim, but it is a Frobenius category.

$$\mathrm{st}(\mathrm{mod}\, kG) := \frac{\mathrm{mod}\ kG}{\mathrm{proj}\ kG} = \frac{\mathrm{mod}\ kG}{\mathrm{inj}\ kG}$$ is the stable module category.

It measures the failure of semisimplicity.

This is a $\boxed{\text{tensor triangulated category}}$ (abbreviated $\otimes$-Δ-ᗢ).

4

THE BALMER SPECTRUM

Commutative algebra:		Tensor triangular geometry:
R ring	algebraic object	$\mathcal{K}$ $\otimes$-Δ-ᗢ
$\rightsquigarrow$	$\rightsquigarrow$	$\rightsquigarrow$
$\mathrm{Spec}(R)$	topological space	$\mathrm{Spc}(\mathcal{K})$

This comes with a universal notion of support that detects thick subcategories.

5

THIS IS USEFUL!

Example: R commutative Noetherian: $\mathrm{Spec}(R) \simeq \mathrm{Spc}(D^{perf}(R)) \simeq \mathrm{Spc}(K^b(\mathrm{proj}\, R))$.

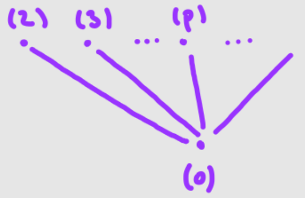
$R = \mathbb{Z}$:

$\mathrm{Spec}(\mathbb{Z}) \simeq \mathrm{Spc}(K^b(\mathrm{proj}\, \mathbb{Z}))$

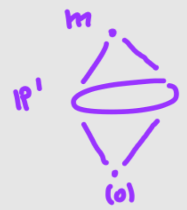
$G = \mathbb{Z}_2 \times \mathbb{Z}_2$:

$\mathrm{Spc}(\mathrm{st}(\mathrm{mod}\, kG)) \simeq \mathrm{Spc}(D^b(\mathrm{mod}\, kG))$

$\mathrm{Spc}(\mathbb{Z}) = \mathrm{Spc}(K^b(\mathrm{proj}\,\mathbb{Z}))$ $\qquad$ $\mathrm{Spc}(\mathrm{st}(\mathrm{mod}\,kG)) = \mathrm{Spc}\left(\frac{\quad}{K^b(\mathrm{proj}\,kG)}\right)$



Lattice:



Max $(H^\bullet(G,k))$:

Proj $(H^\bullet(G,k))$:

Support theories

⑥

The support associated to the Balmer spectrum unifies several notions used in the classification of thick subcategories:

Homotopy theory	Algebraic geometry	Representation theory
[Devinatz, Hopkins, Smith]	[Hopkins, Neeman, Thomason]	[Benson, Carlson, Rickard, Friedlander, Pevtsova]

Support theories

⑦

Depending on the object of interest, they specialize in different homologies:

- G group $\longrightarrow H^\bullet(G,k)$ group cohomology
- A Hopf algebra $\longrightarrow H^\bullet(A,k)$ Hopf cohomology
- A unital associative algebra $\longrightarrow H^\bullet(A,A)$ Hochschild cohomology

If there is a natural subalgebra $B \subset A$, these theories ignore it.

Relative Hochschild cohomology

⑧

Can handle natural subalgebras: for $B \subseteq A$ unital algebras:

$$HH^i_{(A,B)}(A) := \mathrm{Ext}^i_{(A^e,B^e)}(A,A) \quad \text{and} \quad HH^\bullet_{(A,B)}(A) := \bigoplus_{i\in\mathbb{N}} HH^i_{(A,B)}(A).$$

Theorem: 1. $HH^\bullet_{(A,B)}(A)$ is a graded commutative algebra with a cup product.

2. $HH^{\bullet-1}_{(A,B)}(A)$ is a graded Lie algebra.

3. $HH^\bullet_{(A,B)}(A)$ is a Gerstenhaber algebra.

Relative Homological Algebra

(9)

Let $B \subseteq A$ unital subring.

(A,B)-exact:

$$\cdots \longrightarrow M_i \xrightarrow{d_i} M_{i-1} \longrightarrow \cdots$$

(i) $\ker(d_i) = \mathrm{im}(d_{i+1})$ ← A-exact.

(ii) $M_i \cong \ker(d_i) \oplus Q_i$ in mod B.

Equivalently:

$$\begin{tikzcd} \cdots \arrow[r] & M_i \arrow[r, "d_i"] & M_{i-1} \arrow[l, bend left, "s_i"] \arrow[r] & \cdots \end{tikzcd}$$

(1) Over mod B we have:

$d_i d_{i+1} = 0$

$d_{i+1} s_i + s_i d_i = 1_{M_i}$

(2) Over mod B $M_\bullet$ is split exact.

Special Modules

(10)

(A,B)-free: $A \otimes_B X$, X in mod B.

$$\begin{tikzcd} \mathrm{mod}\,A \arrow[r, bend right] \arrow[r, phantom, "\perp"] & \mathrm{mod}\,B \arrow[l, bend right, "A\otimes_B -"'] \end{tikzcd}$$

(A,B)-projective:

$$\begin{tikzcd} & P \arrow[dl, dashed, "h'_A"'] \arrow[d, "h_A"] & \\ M \arrow[r, "g_A"] & N \arrow[l, bend left, "s_B"] \arrow[r] & 0 \end{tikzcd}$$

Bottom row is (A,B)-exact.

(*) **(A,B)-flat:** For every (A,B)-exact $0 \to L \to M \to N \to 0$ then:

$$0 \longrightarrow L \otimes_A F \longrightarrow M \otimes_A F \longrightarrow N \otimes_A F \longrightarrow 0 \quad \text{is } (\mathbb{Z},\mathbb{Z})\text{-exact.}$$

Examples
(11)

$$\begin{array}{ccc}
\text{free} & \Rightarrow & (A,B)\text{-free} \\
\Downarrow & & \Downarrow \\
\text{projective} & \Rightarrow & (A,B)\text{-projective} \\
\Downarrow & & \Downarrow \\
\text{flat} & \Rightarrow & (A,B)\text{-flat}
\end{array}$$

1. $J \subseteq k[x_1, \dots, x_i]$ ideal.

   Not $(k[x_1, \dots, x_i], k)$-flat.

2. $\mathbb{Z}/(n)$ is $(\mathbb{Z}, \mathbb{Z})$-flat but

   not $\mathbb{Z}$-flat.

3. A k-algebra, integral domain, not field.

   Q field of fractions is (A,k)-flat,

   not (A,k)-projective.

Relative Tor
(12)

M a right A-module, N a left A-module.

$$\begin{tikzcd}
(P_\cdot, d_\cdot): \cdots \arrow[r, "d_1"] & P_0 \arrow[l, bend left, "s_0"] \arrow[r, "d_0"] & M \arrow[l, bend left, "s_{-1}"] \arrow[r] & 0
\end{tikzcd}$$

(A,B)-projective resolution.

Truncate at M and apply $- \otimes_A N$.

$$\cdots \xrightarrow{d_2 \otimes 1_N} P_1 \otimes_A N \xrightarrow{d_1 \otimes 1_N} P_0 \otimes_A N \longrightarrow 0$$

$$\mathrm{Tor}_0^{(A,B)}(M,N) := \frac{P_0 \otimes_A N}{\mathrm{im}(d_1 \otimes 1_N)}, \quad \mathrm{Tor}_i^{(A,B)}(M,N) := \frac{\ker(d_i \otimes 1_N)}{\mathrm{im}(d_{i+1} \otimes 1_N)}.$$

Classic results
(13)

Relative Comparison Theorem:

$$\begin{tikzcd}
P_\cdot \arrow[r] \arrow[d, dashed, "f_\cdot"'] & M \arrow[l, bend left] \arrow[r] \arrow[d, "f"] & 0 \\
Q_\cdot \arrow[r] & N \arrow[l, bend left] \arrow[r] & 0
\end{tikzcd}$$

Relative Tor is well-defined and functorial:

$$H_i(P_\cdot \otimes_A N) \cong H_i(P'_\cdot \otimes_A N)$$

Relative Horseshoe Lemma:

$$\begin{tikzcd}
P_\cdot \arrow[r, dashed] \arrow[d] & T_\cdot \arrow[l, dashed, bend left] \arrow[r, dashed] \arrow[d, dashed] & Q_\cdot \arrow[l, dashed, bend left] \arrow[d] \\
{} & {} \arrow[u, dashed] & {}
\end{tikzcd}$$

$$0 \longrightarrow L \rightleftarrows M \rightleftarrows N \longrightarrow 0$$

RELATIVE LONG EXACT SEQUENCE: TOR

14

Theorem: Let $0 \to K \to L \to M \to 0$ be an (A,B)-exact sequence of right A-modules. Then for every left A-module N:

$$\cdots \rightleftarrows \mathrm{Tor}_{i+1}^{(A,B)}(M,N) \to \mathrm{Tor}_i^{(A,B)}(K,N) \rightleftarrows \mathrm{Tor}_i^{(A,B)}(L,N) \rightleftarrows \mathrm{Tor}_i^{(A,B)}(M,N) \longrightarrow \cdots$$

is split exact in 2-out-of-3 terms.

APPLICATION

15

Theorem: (Relative Künneth Theorem) Let $(M_\cdot, m_\cdot)$ be a complex of right A-modules in the relative setting. Let $(N_\cdot, n_\cdot)$ be a complex of left A-modules in the relative setting. Then:

$$\bigoplus_{r+s=i} H_r(M_\cdot) \otimes_A H_s(N_\cdot) \rightleftarrows H_i(M_\cdot \otimes_A N_\cdot) \rightleftarrows \bigoplus_{r+s=i-1} \mathrm{Tor}_1^{(A,B)}(H_r(M_\cdot), H_s(N_\cdot))$$

are split short exact sequences of $\mathbb{Z}$-modules.

APPLICATION

16

The cup product in relative Hochschild cohomology:

$$\cup : HH^n_{(A,B)}(A) \times HH^m_{(A,B)}(A) \longrightarrow HH^{n+m}_{(A,B)}(A)$$

is graded commutative and can be computed via the tensor product of (A,B)-projective resolutions.

17

(A,B)-FLAT

⊛ not the usual definition

For every (A,B)-exact $0 \to L \to M \to N \to 0$ then:

$$0 \to L \otimes_A F \longrightarrow M \otimes_A F \longrightarrow N \otimes_A F \longrightarrow 0 \quad \text{is } (\mathbb{Z},\mathbb{Z})\text{-exact.}$$

Remark:

(A,B)-flat modules preserve (A,B)-exact sequences:

$(M_\cdot, d_\cdot)$ right (A,B)-exact then

$(M_\cdot \otimes_A F, d_\cdot \otimes 1_F)$ is $(\mathbb{Z},\mathbb{Z})$-exact.

Theorem: The following are equivalent:

(1) F is (A,B)-flat.

(2) $\mathrm{Tor}_i^{(A,B)}(M,F) = 0$ for all M and i.

(3) $\mathrm{Tor}_1^{(A,B)}(M,F) = 0$ for all M.

18

APPLICATION

⊛ (A,B)-flat is unusual

Given
$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r] & M \arrow[r] \arrow[l, bend left] & N \arrow[r] \arrow[l, bend left] & 0 \end{tikzcd}$$
(A,B)-exact:

F (A,B)-flat:

$$\begin{tikzcd} 0 \arrow[r] & L \otimes_A F \arrow[r] & M \otimes_A F \arrow[r] \arrow[l, bend left] & N \otimes_A F \arrow[r] \arrow[l, bend left] & 0 \end{tikzcd} \quad \text{is } (\mathbb{Z},\mathbb{Z})\text{-exact.}$$

F "relatively flat": Weibel

$$0 \to L \otimes_A F \longrightarrow M \otimes_A F \longrightarrow N \otimes_A F \longrightarrow 0 \quad \text{is exact.}$$

Proposition: F is (A,B)-flat $\Leftrightarrow$ F is relatively flat.

19

APPLICATION

Proposition: F is (A,B)-flat $\Leftrightarrow$ F is relatively flat.

Proof: Given $0 \to L \rightleftarrows M \rightleftarrows N \to 0$ (A,B)-exact:

$\Rightarrow$) Easy.

$\Leftarrow$) Tor: $\cdots \rightleftarrows \mathrm{Tor}_1^{(A,B)}(N,F) \to L\otimes_A F \underset{r}{\overset{f\otimes 1}{\rightleftarrows}} M\otimes_A F \underset{s}{\overset{g\otimes 1}{\rightleftarrows}} N\otimes_A F \to 0$

Relatively flat: $0 \to L\otimes_A F \xrightarrow{f\otimes 1} M\otimes_A F \xrightarrow{g\otimes 1} N\otimes_A F \to 0$

$\underline{(f\otimes 1)}r\underline{(f\otimes 1)} = \underline{f\otimes 1}$, $(g\otimes 1)s\underline{(g\otimes 1)} = \underline{g\otimes 1}$ $\square$.

Thank you!