The tableaux algebra, Kashiwara crystals, and toric geometry.

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1. The tableaux algebra.

The main object of interest is a very explicitly defined algebra.

__Definition:__ Let $SSYT^n_m$ be the set of semistandard Young tableaux of any shape with at most n rows and entries in $\{1, \ldots, m\}$. Given $T, T' \in SSYT^n_m$ of shapes λ, μ respectively, define $T * T' \in SSYT^n_m$ the semistandard Young tableaux of shape $\lambda + \mu$ obtained by horizontally concatenating the rows and then sorting the entries in each row in weakly increasing order from left to right.

__Example:__

$$\underbrace{\underbrace{\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 3 \\ \hline 4 & \\ \cline{1-1}\end{array}}_{\lambda = (2,2,1)} * \underbrace{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \\ \cline{1-1} 3 & \\ \cline{1-1}\end{array}}_{\mu = (2,1,1)} = \underbrace{\begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 2 \\ \hline 2 & 2 & 3 & \\ \cline{1-3} 3 & 4 & & \\ \cline{1-2}\end{array}}_{\lambda + \mu = (4,3,2)}}_{\text{in } SSYT^3_4}$$

This operation makes $(SSYT^n_m, *, \emptyset)$ into a monoid with very good properties.

__Definition:__ The __tableaux algebra__ ${}_nT_m$ is the monoid algebra of $SSYT^n_m$ over $\Bbbk$ an algebraically closed field of characteristic zero.

Example:

$$
{}_2T_3 = \frac{\Bbbk\left[\begin{array}{|c|}\hline 1\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline\end{array}, \begin{array}{|c|}\hline 3\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\right]}{\left(\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array} * \begin{array}{|c|}\hline 1\\\hline\end{array} - \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} * \begin{array}{|c|}\hline 2\\\hline\end{array}\right)}
$$

Given how explicitly and combinatorially ${}_nT_m$ is defined, we heavily focus on providing explicit and combinatorial proofs for <u>all</u> our results. To begin, we can leverage the good properties of $SSYT^n_m$ for the bookkeeping of some useful properties of ${}_nT_m$.

<u>Theorem</u>: The tableaux algebra ${}_nT_m$ is:

(1) Finitely generated over $\Bbbk$ with the columns of $SSYT^n_m$ forming a generating set.

(2) An integral domain.

(3) Noetherian.

(4) Jacobson (every prime ideal is an intersection of maximal ideals).

(5) Zero Jacobson radical: $J({}_nT_m) = \bigcap\limits_{M \in \operatorname{maxSpec}({}_nT_m)} M = 0$.

Moreover, the product $*$ in ${}_nT_m$ does <u>not</u> arise from an insertion algorithm. The first place we encountered it was when working with combinatorial models for Kashiwara crystals, but we then found it in many other areas, such as toric geometry.

2. Kashiwara crystals.

Intuitively, a crystal is the skeleton of a module of a quantum group. More precisely, given $\mathfrak{g}$ a Lie algebra, denote its associated quantum group by $U_q(\mathfrak{g})$. It is a Hopf algebra with a triangular decomposition (attributed to Lusztig):

$$U_q(\mathfrak{g}) \cong \underbrace{\mathbb{k}[F_\alpha]}_{\substack{\text{negative}\\ \text{degree}}} \otimes \underbrace{\mathbb{k}[K_\alpha^{\pm 1}]}_{\substack{\text{zero}\\ \text{degree}}} \otimes \underbrace{\mathbb{k}[E_\alpha]}_{\substack{\text{positive}\\ \text{degree}}}.$$

For V_λ an irreducible module, we can always find such a decomposition satisfying that the action of F_α, $K_\alpha^{\pm 1}$, E_α is as lowering, diagonal, raising operators, respectively. However, in general there does not exist a PBW basis of $U_q(\mathfrak{g})$ where F_α, $K_\alpha^{\pm 1}$, E_α act as lowering, diagonal, raising operators, for all irreducible modules V_λ simultaneously. This is possible to achieve in the limit $q \to 0$, celebrated work of Kashiwara gives a basis $F, K^{\pm 1}, E$ where F and E act as lowering and raising operators with scalar 1, and $K^{\pm 1}$ act as diagonal operators with scalar 0. This means that for our purposes we can think of crystals as colored graphs, although they carry much more data, such as a weight function and a way of taking tensor products of them. In type A, the three main models used to work with crystals are:

$$\text{string polytopes} \longleftrightarrow \text{Gelfand-Tsetling patterns} \longleftrightarrow \text{semistandard Young tableaux}$$

and there are very explicit ways of moving between these models. Work on string polytopes by Bossinger, Lecouvey, Lenart, Littelman, Patimo, Torres, among many others, developed operations to multiply them, with the goal of finding atomic decompositions of characters and crystals. An important question they are interested in is:

<u>Question:</u> Does the multiplication of two string polytopes coincide with the tensor product of the associated crystals?

We translated their product of string polytopes into our product of semistandard Young tableaux, answered this question in the negative, and obtained new embeddings of crystals similar to the ones in the literature of string polytopes.

<u>Theorem:</u> The map $\begin{aligned} \Phi_T : B(\lambda) &\longrightarrow B(\lambda+\mu) \\ T' &\longmapsto T' * T \end{aligned}$ is a crystal embedding when T is the highest or lowest weight vector of $B(\mu)$.

<u>Proof:</u> For A_μ and Z_μ the highest and lowest weight vectors of $B(\mu)$ we have

$T' * A_\mu = [T' \xleftarrow{col} w_{row}(A_\mu)]$ and $T' * Z_\mu = [T' \xleftarrow{row} w_{col}(Z_\mu)]$, and these insertions are crystal embeddings. This proof also shows why in general $T' * T$ does ~~not~~ come from an insertion algorithm. $\square$.

3. Toric geometry.

For the second application, let's continue studying ${}_nT_m$. Another interesting fact about it is that it is Koszul.

Theorem:

(1) $${}_nT_m = \Bbbk\langle {}_nG_m \rangle \cong \underbrace{\frac{\Bbbk[{}_nE_m]}{{}_n\mathcal{J}_m}}_{\text{part with product relations}} \otimes \underbrace{\Bbbk[{}_nF_m]}_{\text{polynomial part}}, \text{ where}$$

$${}_nG_m := \left\{ T \in SSYT^n_m(1^k) \mid 1 \le k \le n \right\}$$

$${}_nF_n := \left\{ \begin{array}{|c|}\hline n \\ \hline\end{array}\,,\ \begin{array}{|c|}\hline 1 \\ \hline \vdots \\ \hline n-1 \\ \hline\end{array}\,,\ \begin{array}{|c|}\hline 1 \\ \hline \vdots \\ \hline n-1 \\ \hline n \\ \hline\end{array} \right\}$$

$${}_nF_m := \left\{ \begin{array}{|c|}\hline m \\ \hline\end{array}\,,\ \begin{array}{|c|}\hline 1 \\ \hline \vdots \\ \hline n \\ \hline\end{array} \right\} \text{ for } n \neq m$$

$${}_nE_m := {}_nG_m \setminus {}_nF_m.$$

(2) ${}_nT_m$ is a PBW algebra, so it is a Koszul algebra.

This poses two questions.

<u>Question 1:</u> What is the ideal of product relations ${}_n\mathcal{B}_m$?

<u>Question 2:</u> What is the size of a minimal generating set of ${}_n\mathcal{B}_m$?

Both questions are related to toric geometry. Beginning with the first question:

<u>Theorem:</u> The zero locus $\mathcal{V}({}_n\mathcal{B}_m)$ is a toric variety.

<u>Proof:</u> Since ${}_n\mathcal{B}_m$ is a prime binomial ideal, the result follows from work of Eisenbud and Sturmfels. $\square$.

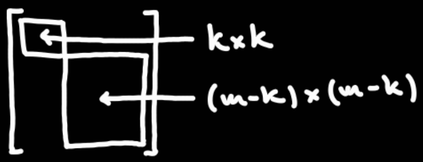
In fact, the diagonal (or single-indexed) tableaux algebra is isomorphic to the Gelfand-Tsetling semigroup. In turn, the Gelfand-Tsetling semigroup is a flat degeneration of the Plücker algebra:

$$ {}_n T_n \cong \mathcal{G}T_n \cong \text{Plücker algebra} \Big/ \left(\begin{array}{l} \text{initial ideal of the ideal} \\ \text{of Plücker relations.} \end{array} \right) $$

so we seeked for (and achieved) a similar result for ${}_n T_m$. Indeed:

<u>Theorem:</u> The tableaux algebra ${}_n T_m$ is a flat degeneration of the algebra of global sections $\bigoplus_{\underline{a}} \Gamma(SL_m/{}_nQ_m, L^{\underline{a}})$. The zero locus $\mathcal{V}({}_n\mathcal{B}_m)$ is a toric degeneration of the partial flag variety $SL_m/{}_nQ_m$.

Here $n \neq m$, $nQ_m := \bigcap_{k=1}^{n} P_k$ with $P_k \subsetneq SL_m$ the maximal parabolic



, and $L^{\underline{a}}$ is the tensor product of the ample generators of $\mathrm{Pic}(SL_m/P_k)$.

The proof heavily relies on work of Gonciulea, Hibi, and Lakshmibai. As for the second question:

<u>Theorem:</u>

(1) We explicitly construct a minimal generating set of the relations $n\mathcal{I}_m$:

$$nB_m := \{T \cdot T' - L_{T,T'} \cdot R_{T,T'} \mid T, T' \in nG_m\}.$$

(2) We provide an explicit formula for $|nB_m|$ that exclusively depends on n and m.

As a consequence of the two theorems above (leveraging toric geometry), nB_m is a Gröbner basis of $n\mathcal{I}_m$. Thus, we have in fact counted the minimal number of Plücker relations in SL_m/nQ_m, the incidence relations between different Grassmannians in SL_m/nQ_m, and the minimal number of Plücker-Grassmann relations within $Gr(i,m)$.

All these formulas are new, and for the first two cases, the only known ones (we do <u>not</u> use that # Plücker = # incidence + # Plücker-Grassmann).