THE ZIEGLER SPECTRUM OF A DISCRETE VALUATION RING

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Abstract. This note gives a description of the Ziegler spectrum of a discrete valuation ring, emphasizing the interconnection between logic and representation theory.

1. Preliminaries

Definition 1.1. A *discrete valuation ring*, denoted D, is a principal ideal domain with exactly one non-zero maximal ideal, denoted $D\pi$ for a fixed $\pi \in D$.

Equivalently, D is a local ring, a principal ideal domain, and not a field. Equivalently, D is a principal ideal domain with a unique non-zero prime ideal. In all cases, D is commutative by definition. Unless explicitly stated otherwise, all modules in this note will be over D.

Example 1.2. The following are modules.

(1) The field of fractions Q of D.
(2) The ideals $D\pi^n$ for $n \in \mathbb{N}$.
(3) The quotients $D/D\pi^n$ for $n \in \mathbb{N}$.
(4) The ring of power series $D[\![x]\!]$.

The ideals of D form a descending filtration of D:

$$D = D\pi^0 \supseteq D\pi \supseteq D\pi^2 \supseteq \cdots \supseteq D\pi^n \supseteq \cdots \supseteq 0, \tag{1.3}$$

which can be assembled into the following sequence of injections

$$0 \longrightarrow \tfrac{D}{D\pi} \hookrightarrow \tfrac{D}{D\pi^2} \hookrightarrow \cdots \hookrightarrow \tfrac{D}{D\pi^n} \hookrightarrow \cdots \hookrightarrow \varinjlim_{n\in\mathbb{N}} \tfrac{D}{D\pi^n} \tag{1.4}$$

and the following sequence of surjections

$$\varprojlim_{n\in\mathbb{N}} \tfrac{D}{D\pi^n} \twoheadrightarrow \cdots \twoheadrightarrow \tfrac{D}{D\pi^n} \twoheadrightarrow \cdots \twoheadrightarrow \tfrac{D}{D\pi^2} \twoheadrightarrow \tfrac{D}{D\pi} \longrightarrow 0. \tag{1.5}$$

Definition 1.6. The *Prüfer module of D*, denoted $D[\pi^\infty]$, is the direct limit of $D/D\pi^n$ for $n \in \mathbb{N}$.

$$D[\pi^\infty] \coloneqq \varinjlim_{n\in\mathbb{N}} \frac{D}{D\pi^n} \tag{1.7}$$

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Definition 1.8. The π-*completion of* D, denoted $\widehat{D}$, is the inverse limit of $D/D\pi^n$ for $n \in \mathbb{N}$.

$$\widehat{D} \coloneqq \varprojlim_{n\in\mathbb{N}} \frac{D}{D\pi^n} \tag{1.9}$$

Observe that the Prüfer module of D fits into the following short exact sequence

$$0 \longrightarrow D \hookrightarrow Q \twoheadrightarrow D[\pi^\infty] \longrightarrow 0, \tag{1.10}$$

so in particular $D[\pi^\infty] = \operatorname{coker}(D \hookrightarrow Q) \cong Q/D \equiv \{u/\pi^n \mid u \notin D\pi \text{ and } n \in \mathbb{N}_{>0}\}$. Similarly, the π-completion of D fits into the following short exact sequence

$$0 \longrightarrow D[\![x]\!](x-\pi) \hookrightarrow D[\![x]\!] \twoheadrightarrow \widehat{D} \longrightarrow 0, \tag{1.11}$$

so in particular $\widehat{D} = \operatorname{coker}(D[\![x]\!](x-\pi) \hookrightarrow D[\![x]\!]) \cong D[\![x]\!]/D[\![x]\!](x-\pi)$, the formal power series that are divisible by $(x-\pi)$. As we will justify below, $D[\pi^\infty]$ is the injective hull of $D/D\pi$ and $\widehat{D}$ is the pure-injective hull of D.

2. The points of $\mathrm{Zg}(D)$

Definition 2.1. The set of points of the *Ziegler spectrum* of D, denoted $\mathrm{Zg}(D)$, is the set of isomorphism classes of indecomposable pure-injective modules over D.

To determine the Ziegler spectrum of D, we recall some definitions and results concerning purity.

Definition 2.2. Let $f : M \to N$ be an injective map of modules. We say that f is *pure* when for every finite system of equations with coefficients in D and constant terms from M, if there is a solution in N then there is a solution in M.

Denote a system of equations with coefficients in D, variables $\vec{x} = [x_1, \ldots, x_k]$, and constant term $\vec{l} \in L^k$ by $\theta(\vec{x}, \vec{l})$. Denote the set of solutions $\{\vec{m} \in M^k \mid \theta(\vec{m}, \vec{l})\}$ by $\theta(M, \vec{l})$. An injective map $f : M \to N$ is pure when given $\theta(\vec{x}, \vec{m})$ such that $\theta(N, f(\vec{m})) \neq \emptyset$ then $\theta(M, \vec{m}) \neq \emptyset$, where $f(\vec{m})$ is understood to be the application of f to each entry of $\vec{m}$.

Proposition 2.3. *Let $f : M \to N$ be an injective map of modules. The following are equivalent.*

(1) $f : M \to N$ is pure.

(2) $f \otimes_D \mathrm{id}_L : M \otimes_D L \to N \otimes_D L$ is an injective map of Abelian groups for all L finitely presented module.

Example 2.4. There is a natural pure embedding $D \to \widehat{D}$. To see this, note first that taking the inverse limit of the inverse system of short exact sequences

$$\{\ 0 \longrightarrow D\pi^n \hookrightarrow D \twoheadrightarrow D/D\pi^n \longrightarrow 0\ \}_{n\in\mathbb{N}} \tag{2.5}$$

gives the exact sequence

$$\begin{array}{ccccccc} 0 \longrightarrow & \varprojlim_{n\in\mathbb{N}} D\pi^n & \hookrightarrow & \varprojlim_{n\in\mathbb{N}} D & \longrightarrow & \varprojlim_{n\in\mathbb{N}} D/D\pi^n \\ & \downarrow \cong & & \downarrow \cong & & \downarrow = \\ & 0 & \longrightarrow & D & \longrightarrow & \widehat{D} \end{array} \tag{2.6}$$

because the inverse limit is a left exact functor. This natural map $D \to \widehat{D}$ is thus injective, and it coincides with the natural map $D \to \widehat{D} \otimes_D D$. A more elaborate argument shows that $L \to \widehat{D} \otimes_D L$

is injective for any finitely presented module L. Since $D \otimes_D L$ is a finitely presented module when L is a finitely presented module, then

$$D \otimes_D L \longrightarrow \widehat{D} \otimes_D D \otimes_D L \cong \widehat{D} \otimes_D L \tag{2.7}$$

is injective. Thus $D \to \widehat{D}$ is a pure embedding by Proposition 2.3.

Definition 2.8. Let P be a module, let $f : M \to N$ be any pure embedding of modules, and let $g : M \to P$ be any map of modules. We say that M is *pure-injective* when there exists a map $h : N \to P$ such that $hf = g$.

$$\begin{array}{ccccc} 0 & \longrightarrow & M & \overset{f}{\hookrightarrow} & N \\ & & {\scriptstyle g}\downarrow & \swarrow{\scriptstyle h} & \\ & & P & & \end{array} \tag{2.9}$$

Example 2.10.

(1) The module $D[\pi^\infty]$ is pure-injective because it is injective. In fact, it is the injective hull of $D/D\pi$ because

$$D[\pi^\infty] = \varinjlim_{n\in\mathbb{N}} \frac{D}{D\pi^n} = \frac{Q}{D} = \frac{D_\pi}{D} = \operatorname{InjHull}\left(\frac{D}{D\pi}\right). \tag{2.11}$$

(2) The module $\widehat{D}$ is pure-injective. We will use the following facts. First, Artinian modules are linearly compact. Second, the inverse limit of linearly compact modules is linearly compact. Third, over a commutative ring being linearly compact is equivalent to being algebraically compact. Fourth, being algebraically compact is equivalent to being pure-injective. Now, $\widehat{D}$ is algebraically compact because it is the inverse limit of the Artinian modules $D/D\pi^n$ for $n \in \mathbb{N}_{>0}$. Thus $\widehat{D}$ is pure-injective. In fact, it is the pure-injective hull of D.

(3) Any module of finite endolength (that is, of finite length over its endomorphism ring) is pure-injective. This can be proven by a detour through algebraic compactness.

Theorem 2.12. *The indecomposable pure-injective modules of D are:*

(1) $D/D\pi^n$, for $n \in \mathbb{N}_{>0}$.
(2) $D[\pi^\infty]$, the Prüfer module of D.
(3) $\widehat{D}$, the π-adic completion of D.
(4) Q, the field of fractions of D.

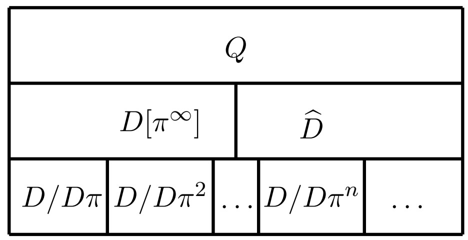
The quotients $D/D\pi^n$ for $n \in \mathbb{N}_{>0}$ are Artinian rings and $\operatorname{End}_D(D/D\pi^n) \cong D/D\pi^n$, so $D/D\pi^n$ is of finite endolength, which by Example 2.10 implies that $D/D\pi^n$ are pure-injective. Example 2.10 justifies that $\widehat{D}$ is pure-injective, and that $D[\pi^\infty]$ is injective. The field of fractions Q is also injective as a module because of Baer's criterion (in fact, Q is the injective hull of D). Inspecting Definition 2.8 shows that all injective modules are pure-injective modules, so $D[\pi^\infty]$ and Q are pure-injective. The ideal $D\pi$ is maximal, so π is prime in D, so the quotients $D/D\pi^n$ for $n \in \mathbb{N}_{>0}$ are indecomposable. The submodules of $D[\pi^\infty] \cong Q/D$ are

$$\frac{\frac{1}{\pi^n}D}{D} = \left\{ \frac{u}{\pi^n} \mid u \notin D\pi \right\} \tag{2.13}$$

for $n \in \mathbb{N}_{>0}$, so it is indecomposable. The ideal $D[\![x]\!](x-\pi) \subsetneq D[\![x]\!]$ is prime, so the quotient $\widehat{D} \cong D[\![x]\!]/D[\![x]\!](x-\pi)$ is indecomposable. Since D is a commutative domain, its field of fractions Q is indecomposable as a module.

We can now draw the points of the Ziegler spectrum of D as follows.

$\mathrm{Zg}(D)$



The reason for this arrangement is that it captures the Cantor–Bendixon rank of the points. Namely, Q has Cantor–Bendixon rank 2, $D[\pi^\infty]$ and $\widehat{D}$ both have Cantor–Bendixon rank 1, and each of the $D/D\pi^n$ for $n \in \mathbb{N}_{>0}$ has Cantor–Bendixon rank 0.

Similarly, the length and endolength of these modules will also give topological information about the picture. The points Q, $D[\pi^\infty]$, and $\widehat{D}$ do not have finite length, and for $n \in \mathbb{N}_{>0}$ the length of $D/D\pi^n$ is n. The endolength of Q is finite, $D[\pi^\infty]$ and $\widehat{D}$ do not have finite endolength, and for $n \in \mathbb{N}_{>0}$ the endolength of $D/D\pi^n$ is n.

3. The topology of $\mathrm{Zg}(D)$

Definition 3.1. A *positive primitive condition*, denoted ϕ, is a logical condition having the form

$$\exists x_{k+1}, \dots, x_n \bigwedge_{j=1}^{m} \sum_{i=1}^{n} x_i d_{ij} = 0 \tag{3.2}$$

where $k, m, n \in \mathbb{N}$ are fixed, $\{x_i\}_{i=k+1,\dots,n} \subseteq M$ for some module M, and $\{d_{ij}\}_{i=1,\dots,n}^{j=1,\dots,m} \subseteq D$. We denote by $\phi(M)$ the set of solutions to the above system of equations. Namely

$$\phi(M) = \left\{ [x_1, \dots, x_k] \in M^k \mid \exists x_{k+1}, \dots, x_n \in M \text{ with } \bigwedge_{j=1}^{m} \sum_{i=1}^{n} x_i d_{ij} = 0 \right\}. \tag{3.3}$$

Observe that the set of solutions $\phi(M)$ is an Abelian group.

Example 3.4. The following are positive primitive conditions.

(1) The system of equations $\theta(\vec{x}, \vec{l})$ of Definition 2.2.
(2) The condition $Ix = 0$ for I an ideal of D. Writing $I = (g_1, \dots, g_n)$ for some generators $g_1, \dots, g_n$, this condition is $\wedge_{i=1}^{n} x g_i = 0$.
(3) The condition $I \mid x$ for I an ideal of D. Writing $I = (g_1, \dots, g_n)$ as before, this condition is $\exists x_1, \dots, x_n \, (x = \sum_{i=1}^{n} x_i g_i)$.

Definition 3.5. The following sets form a basis of opens of $\mathrm{Zg}(D)$.

$$\left(\frac{\phi}{\psi} \right) = \{ N \in \mathrm{Zg}(D) \mid \phi(N) \gneq \psi(N) \} \tag{3.6}$$

This is equivalent to the usual definition giving the closed sets of $\mathrm{Zg}(D)$ in terms of definable subcategories. Namely, letting $\mathcal{X}$ be a definable subcategory of modules, the closed sets of $\mathrm{Zg}(D)$ are the set of isomorphism classes of indecomposable pure-injective modules in $\mathcal{X}$.

Example 3.7. The following open sets contain each of the points of $\mathrm{Zg}(D)$.

(1) $D/D\pi^n$ for $n \in \mathbb{N}_{>0}$ is contained in

$$\frac{(D\pi^{n-1} \mid x) \wedge (D\pi x = 0)}{(D\pi^n \mid x) \wedge (D\pi x = 0)} = \left\{ \frac{D}{D\pi^n} \right\}, \tag{3.8}$$

making it an isolated point.

(2) $D[\pi^\infty]$ is contained in

$$\frac{(D\pi x = 0)}{(x = 0)} = \{D[\pi^\infty]\} \cup \left\{ \frac{D}{D\pi^n} \mid n \in \mathbb{N}_{>0} \right\}, \tag{3.9}$$

which has a neighborhood basis of opens formed by

$$\frac{(D\pi^{n+1} x = 0)}{(D\pi^n x = 0)} = \{D[\pi^\infty]\} \cup \left\{ \frac{D}{D\pi^m} \mid m \in \mathbb{N}_{>n} \right\} \tag{3.10}$$

for all $n \in \mathbb{N}_{>0}$.

(3) $\widehat{D}$ is contained in

$$\frac{(x = x)}{(D\pi \mid x)} = \left\{ \widehat{D} \right\} \cup \left\{ \frac{D}{D\pi^n} \mid n \in \mathbb{N}_{>0} \right\}, \tag{3.11}$$

which has a neighborhood basis of opens formed by

$$\frac{(D\pi^n \mid x)}{(D\pi^{n+1} \mid x)} = \left\{ \widehat{D} \right\} \cup \left\{ \frac{D}{D\pi^m} \mid m \in \mathbb{N}_{>n} \right\} \tag{3.12}$$

for all $n \in \mathbb{N}_{>0}$.

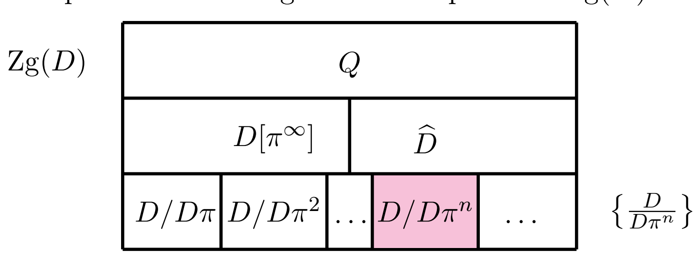
(4) Q is contained in the closure of both $D[\pi^\infty]$ and $\widehat{D}$. Thus all open neighborhoods of Q must contain both $D[\pi^\infty]$ and $\widehat{D}$, and can only exclude finitely many of the $D/D\pi^n$ for $n \in \mathbb{N}_{>0}$. A neighborhood basis of opens is formed by

$$\frac{(x = x)}{(D\pi^n x = 0)} = \mathrm{Zg}(D) \setminus \left\{ \frac{D}{D\pi^m} \mid m \in \mathbb{N}_{<n+1} \right\} = \frac{(D\pi^n \mid x)}{(x = 0)} \tag{3.13}$$

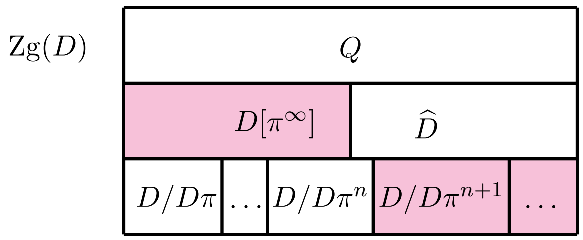
for all $n \in \mathbb{N}_{>0}$.

Example 3.14. The closed points of $\mathrm{Zg}(D)$ are exactly Q and $D/D\pi^n$ for $n \in \mathbb{N}_{>0}$.

We now draw a typical open set containing each of the points of $\mathrm{Zg}(D)$. The opens (3.8) look like

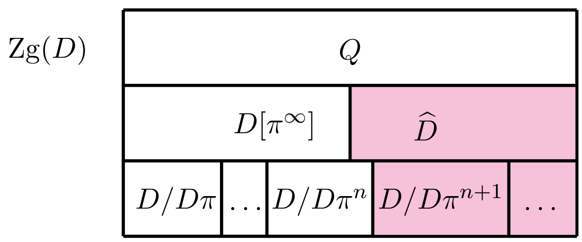


the opens (3.10) look like



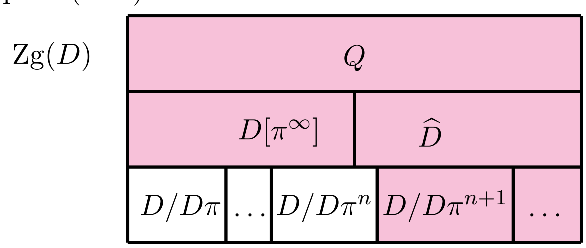
$\{D[\pi^\infty]\} \cup \left\{ \frac{D}{D\pi^m} \mid m \in \mathbb{N}_{>n} \right\}$

the opens (3.12) look like



$\left\{ \widehat{D} \right\} \cup \left\{ \frac{D}{D\pi^m} \mid m \in \mathbb{N}_{>n} \right\}$

and the opens (3.13) look like



$\mathrm{Zg}(D) \setminus \left\{ \frac{D}{D\pi^m} \mid m \in \mathbb{N}_{<n+1} \right\}$

.

Corollary 3.15. *The Ziegler spectrum of D is compact.*

Proof. Let $C = \{U_i\}_{i \in I}$ be an open cover of $\mathrm{Zg}(D)$. If $\mathrm{Zg}(D) \in C$ we are done, so suppose $\mathrm{Zg}(D) \notin C$. Since C covers Q, it contains an open set of the form (3.13), say $\mathrm{Zg}(D) \setminus \{D/D\pi^m \mid m \in \mathbb{N}_{<n+1}\}$ for a fixed $n \in \mathbb{N}_{>0}$. This covers all $\mathrm{Zg}(D)$ except $D/D\pi, \dots, D/D\pi^n$. Since C covers these points, for each $m = 1, \dots, n$ there must be an open set $U_{i_m} \in C$ such that $\{D/D\pi^m\} \in U_{i_m}$. Then $\{\mathrm{Zg}(D) \setminus \{D/D\pi^m \mid m \in \mathbb{N}_{<n+1}\}, U_{i_1}, \dots, U_{i_m}\}$ is a finite subcover of C. □

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